
Stability Results and Algorithmic Strategies for the Finite Element Approach to the Immersed Boundary Method.

Daniele Boffi¹, Lucia Gastaldi², and Luca Heltai³

¹ Dipartimento di Matematica "F. Casorati", Via Ferrata 1, I-27100 Pavia, Italy
daniele.boffi@unipv.it

² Dipartimento di Matematica, Via Valotti 9, I-25133 Brescia, Italy
gastaldi@ing.unibs.it

³ Dipartimento di Matematica "F. Casorati", Via Ferrata 1, I-27100 Pavia, Italy
luca.heltai@unipv.it

Summary. The immersed boundary method is both a mathematical formulation and a numerical method for the study of fluid structure interactions. Many numerical schemes have been introduced to reduce the difficulties related to the non-linear coupling between the structure and the fluid evolution, however numerical instabilities arise when explicit or semi-implicit methods are considered. In this work we present a stability analysis based on energy estimates for the variational formulation of the immersed boundary method.

A two dimensional incompressible fluid and a boundary in the form of a simple closed curve are considered. We use a linearization of the Navier-Stokes equations and a linear elasticity model to prove the unconditional stability of the fully implicit discretization, achieved with the use of a backward Euler method for both the fluid and the structure evolution (BE/BE), and we present a computable CFL condition for the semi-implicit method where the fluid terms are treated implicitly while the structure is treated explicitly (FE/BE).

Key words: immersed boundary method; finite element method; numerical stability; CFL condition; fluid structure interaction.

1 Introduction

The idea behind the immersed boundary (IB) method lies on the observation that the Navier-Stokes equations for incompressible fluids express nothing more than Newton's law $\mathbf{F} = m\mathbf{a}$ in an Eulerian and "fluid-specialized" framework. The IB method consists in adding to the Navier-Stokes equations some additional "internal" forces concentrated on the particles of the "fluid-solid" material to compensate the fluid behavior with the missing elastic part, in order to simulate efficiently the interaction between a fluid and an elastic material.

In this paper we will present a numerical analysis of the stability of the IB method applied to a one-dimensional volume-less and massless membrane, immersed in a two-dimensional fluid domain, modeled by the dynamic Stokes equations.

Numerical instabilities arise when computations are carried on using semi-implicit or explicit time-stepping techniques which require a careful choice of the discretization parameters. We will address the stability aspect of IB computations taking advantage of the natural energy estimates that arise from the use of a variational approach to the IB method, as introduced in [2, 3].

In Sect. 2 we briefly present the finite element IB method, as it was introduced in [2, 3] and the elasticity model that will be used throughout the paper. Section 3 describes the time stepping schemes that will be analyzed in Sect. 4, while Sect. 5 is dedicated to numerical validation and conclusions.

2 The Finite Element Immersed Boundary Method

Let Ω be a two dimensional domain containing both the fluid and the elastic membrane. To be more precise, for all $t \in [0, T]$, let Γ_t be a simple closed elastic curve, the configuration of which is given in a parametric form, $\mathbf{X}(s, t)$, $0 \leq s \leq L$, $\mathbf{X}(0, t) = \mathbf{X}(L, t)$, where the parameter s marks a material point and L is related to the unstressed length of the boundary.

$\mathbf{X}(s, t)$ represents the position in Ω of the material point which was labeled by s at the initial time. We are interested in expressing formally the force exerted by the structure on the fluid in terms of the elastic force density $\mathbf{f}(s, t)$ generated by the deformation of the immersed material itself. In the IB method this is achieved by mean of the defining properties of the Dirac delta distribution δ :

$$\mathbf{F}(\mathbf{x}, t) = \int_D \mathbf{f}(s, t) \delta(\mathbf{x} - \mathbf{X}(s, t)) ds, \quad \text{in } \Omega \times]0, T[. \quad (1)$$

Here the Dirac delta is used as a way to pass from the Lagrangian to the Eulerian formulation by introducing an “implicit” change of variables.

The force generated by the element of boundary ds on the fluid is $\mathbf{f}(s, t) ds$. We will concentrate on the case of linearized hyper-elastic incompressible materials, characterized by the existence of a positive potential energy density Ψ associated with the deformation of the elastic material and which is independent on translations and (linearized) rotations of the material itself.

The relation between the potential energy density Ψ and the force \mathbf{f} is given through the use of the deformation tensor \mathbb{F} and the first Piola-Kirchoff stress tensor \mathbb{P} as follows:

$$\mathbb{F}_{ij} := \frac{\partial \mathbf{X}_j(s, t)}{\partial s_i}, \quad \mathbb{P}_{ij}(s, t) = \frac{\partial \Psi}{\partial \mathbb{F}_{ij}}(s, t), \quad \mathbf{f}_j(s, t) = \frac{\partial \mathbb{P}_{ij}}{\partial s_i}(s, t),$$

where, in the last equation, summation is implied over repeated indices.

In the two dimensional case $i = 1$ and $j = 1, 2$, therefore all tensors become vectors. We will use a linear “fiber-like” formulation where the Piola-Kirchoff stress

tensor is defined by a scalar tension $T = \kappa|\mathbb{F}|$ and the versor $\tau = \mathbb{F}/|\mathbb{F}|$ tangent to the immersed curve, therefore the following hold

$$\mathbb{P} = T\tau = \kappa\mathbb{F}, \quad \mathbf{f} = \kappa \frac{\partial^2 \mathbf{X}}{\partial s^2}, \quad \Psi(\mathbb{F}) = \frac{\kappa}{2} |\mathbb{F}|^2, \quad (2)$$

where κ is the elasticity constant of the material along the immersed boundary.

We observe that, in equation (1), the boundary force \mathbf{f} is multiplied by a two dimensional Dirac distribution, over a domain of dimension one, so that the resulting force density \mathbf{F} is a one dimensional Dirac distribution along Γ_t and the following Lemma holds true.

Lemma 1. *Assume that, for all $t \in [0, T]$, the immersed boundary Γ_t is Lipschitz continuous and that $\mathbf{f} \in L^2([0, L] \times]0, T[)$. Then for all $t \in]0, T[$, the force density $\mathbf{F}(t)$, defined formally in (1), is a distribution belonging to $H^{-1}(\Omega)^2$ defined as follows: for all $\mathbf{v} \in H_0^1(\Omega)^2$*

$$\langle \mathbf{F}(t), \mathbf{v} \rangle_{H_0^{-1}} = \int_0^L \mathbf{f}(s, t) \cdot \mathbf{v}(\mathbf{X}(s, t)) \, ds \quad \forall t \in]0, T[. \quad (3)$$

Let \mathcal{T}_h be a subdivision of Ω into triangles or rectangles. We denote by h_x the biggest diameter of the elements of \mathcal{T}_h . We then consider two finite dimensional subspaces $\mathbf{V}_h \subseteq H_0^1(\Omega)^2$ and $Q_h \subseteq L_0^2(\Omega)$. It is well known that the pair of spaces \mathbf{V}_h and Q_h need to satisfy the inf-sup condition in order to have existence, uniqueness and stability of the discrete solution of the Navier-Stokes problem (see [4]).

Next, let $s_i, i = 0, \dots, m$ with $s_0 = 0$ and $s_m = L$, be $m + 1$ distinct points of the interval $[0, L]$. We set $h_s = \max_{0 \leq i \leq m} |s_i - s_{i-1}|$. Let \mathbf{S}_h be the finite element space of piecewise linear vectors defined on $[0, L]$ as follows

$$\mathbf{S}_h = \left\{ \mathbf{Y} \in C^0([0, L]; \Omega) : \mathbf{Y}|_{[s_{i-1}, s_i]} \in \mathcal{P}^1([s_{i-1}, s_i])^2, i = 1, \dots, m, \right. \\ \left. \mathbf{Y}(s_0) = \mathbf{Y}(s_m) \right\} \quad (4)$$

where $\mathcal{P}^1(I)$ stands for the space of affine polynomials on the interval I . For an element $\mathbf{Y} \in \mathbf{S}_h$ we shall use also the following notation $\mathbf{Y}_i = \mathbf{Y}(s_i)$ for $i = 0, \dots, m$.

Taking into account Lemma 1, it is possible to show that

$$\langle \mathbf{F}_h(t), \mathbf{v} \rangle = \sum_{i=0}^{m-1} \kappa \left(\frac{\partial \mathbf{X}_{h, i+1}}{\partial s}(t) - \frac{\partial \mathbf{X}_{h, i}}{\partial s}(t) \right) \mathbf{v}(\mathbf{X}_{h, i}(t)). \quad (5)$$

Notice that the right hand side of (5) is meaningful, since \mathbf{v} is continuous as it is required for the elements in \mathbf{V}_h .

The finite element discretization of the IB method reads:

Problem 1. Given $\mathbf{u}_{0h} \in \mathbf{V}_h$ and $\mathbf{X}_{h,0} \in \mathbf{S}_h$, for all $t \in]0, T[$, find $(\mathbf{u}_h(t), p_h(t)) \in \mathbf{V}_h \times Q_h$ and $\mathbf{X}_h(t) \in \mathbf{S}_h$, such that

$$\rho \frac{d}{dt}(\mathbf{u}_h(t), \mathbf{v}) + \mu(\nabla \mathbf{u}_h(t), \nabla \mathbf{v}) - (\nabla \cdot \mathbf{v}, p_h(t)) = \langle \mathbf{F}_h(t), \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{V}_h \quad (6)$$

$$(\nabla \cdot \mathbf{u}_h(t), q) = 0 \quad \forall q \in Q_h \quad (7)$$

$$\langle \mathbf{F}_h(t), \mathbf{v} \rangle = \sum_{i=0}^{m-1} \kappa \left(\frac{\partial \mathbf{X}_{h^{i+1}}}{\partial s}(t) - \frac{\partial \mathbf{X}_{h^i}}{\partial s}(t) \right) \mathbf{v}(\mathbf{X}_{h^i}(t)) \quad \forall \mathbf{v} \in \mathbf{V}_h \quad (8)$$

$$\frac{\partial \mathbf{X}_{h^i}}{\partial t}(t) = \mathbf{u}_h(\mathbf{X}_{h^i}(t), t) \quad \forall i = 0, 1, \dots, m \quad (9)$$

$$\mathbf{u}_h(\mathbf{x}, 0) = \mathbf{u}_{0h}(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega \quad (10)$$

$$\mathbf{X}_{h^i}(0) = \mathbf{X}_0(s_i) \quad \forall i = 1, \dots, m. \quad (11)$$

3 Time Discretization by Finite Differences

In [8] it was shown how a fully implicit discretization in time for both the elasticity and the fluid equations appears to be unconditionally stable. We will not report numerical experiments on this approach (referred in the sequel as the Backward Euler/Backward Euler, or BE/BE scheme), we will however show that this approach is unconditionally stable.

A natural alternative to the fully implicit method is the use of a semi-implicit modification. We will refer to this time stepping technique, which couples the pressure and diffusion implicitly in a Stokes solve while treating the elastic terms explicitly, as the Forward Euler/Backward Euler (in short FE/BE) scheme, following the notations of [6]. We will show that this method is not unconditionally stable and we will give an appropriate CFL condition needed for it to remain stable.

Let Δt denote the time step and let us indicate by the superscript n an unknown function at time $t_n = n\Delta t$, so that the number of time steps needed to reach the final time T is N .

The two schemes can be formally described in a unified way:

Problem 2. Given $\mathbf{u}_{0h} \in \mathbf{V}_h$ and $\mathbf{X}_{0h} \in \mathbf{S}_h$, set $\mathbf{u}_h^0 = \mathbf{u}_{0h}$ and $\mathbf{X}_h^0 = \mathbf{X}_{0h}$, then for $n = 0, 1, \dots, N-1$

Step 1. compute the source term

$$\langle \mathbf{F}_h^{n+1}, \mathbf{v} \rangle = \sum_{i=0}^{m-1} \kappa \left(\frac{\partial \mathbf{Y}_{i+1}}{\partial s} - \frac{\partial \mathbf{Y}_i}{\partial s} \right) \mathbf{v}(\mathbf{Y}_i) \quad \forall \mathbf{v} \in \mathbf{V}_h;$$

Step 2. find $(\mathbf{u}_h^{n+1}, p_h^{n+1}) \in \mathbf{V}_h \times Q_h$, such that

$$\begin{aligned} \rho \left(\frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t}, \mathbf{v} \right) + \mu(\nabla \mathbf{u}_h^{n+1}, \nabla \mathbf{v}) - (\nabla \cdot \mathbf{v}, p_h^{n+1}) \\ = \langle \mathbf{F}_h^{n+1}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{V}_h \\ (\nabla \cdot \mathbf{u}_h^{n+1}, q) = 0 \quad \forall q \in Q_h \end{aligned}$$

Step 3. find $\mathbf{X}_h^{n+1} \in \mathbf{S}_h$, such that

$$\frac{\mathbf{X}_{hi}^{n+1} - \mathbf{X}_{hi}^n}{\Delta t} = \mathbf{u}_h^{n+1}(\mathbf{Y}_i^n) \quad \forall i = 1, \dots, m,$$

where \mathbf{Y} is \mathbf{X}_h^n in the FE/BE scheme and \mathbf{X}_h^{n+1} in the BE/BE one.

The FE/BE scheme introduced in Problem 2 is computable, while the BE/BE scheme (here introduced only formally) requires the implementation of some sort of iterative scheme. We refer to [8] for the derivation of one such a scheme and we will only give some theoretical results about its unconditional stability.

4 Stability Analysis by Energy Estimates

We prove here the unconditional stability of the fully implicit method, and present the CFL conditions that need to be satisfied to preserve the stability of the semi-implicit numerical scheme.

In the following we will make extensive use of the total potential energy of the elastic material, which in our case is defined as

$$E[\mathbf{X}(t)] := \int_D \Psi(\mathbb{F}(s, t)) ds = \frac{\kappa}{2} \left\| \frac{\partial \mathbf{X}(t)}{\partial s} \right\|_{0,D}^2. \quad (12)$$

4.1 Stability of the continuous problem

The following stability estimate holds true for the solution of both the continuous and the space discretized problem:

Lemma 2. For $t \in]0, T[$, let $\mathbf{u}_h(t) \in \mathbf{V}_h$, $p_h(t) \in Q_h$ and $\mathbf{X}_h(t) \in \mathbf{S}_h$ be a solution of Problem 1, then it holds:

$$\frac{\rho}{2} \frac{d}{dt} \|\mathbf{u}_h(t)\|_{0,\Omega}^2 + \mu \|\nabla \mathbf{u}_h(t)\|_{0,\Omega}^2 + \frac{\kappa}{2} \frac{d}{dt} \left\| \frac{\partial \mathbf{X}_h(t)}{\partial s} \right\|_{0,D}^2 = 0. \quad (13)$$

Using the same principles it is possible to provide some stability results also for the fully discretized case both in the BE/BE case,

Theorem 1 Let $\mathbf{u}_h, \mathbf{X}_h$ be a solution of the BE/BE scheme in Problem 2.

The following discrete energy inequality holds:

$$\begin{aligned} & \frac{\rho}{2\Delta t} (\|\mathbf{u}_h^{n+1}\|_{0,\Omega}^2 - \|\mathbf{u}_h^n\|_{0,\Omega}^2) + \mu \|\nabla \mathbf{u}_h^{n+1}\|_{0,\Omega}^2 \\ & + \frac{\kappa}{2\Delta t} \left(\left\| \frac{\partial \mathbf{X}_h^{n+1}}{\partial s} \right\|_{0,D}^2 - \left\| \frac{\partial \mathbf{X}_h^n}{\partial s} \right\|_{0,D}^2 \right) \leq 0, \end{aligned} \quad (14)$$

as well as in the FE/BE case:

Theorem 2 Let $\mathbf{u}_h^n, \mathbf{X}_h^n$ be a solution at time $t = n\Delta t$ of the FE/BE scheme of Problem 2 and let L^n be defined as

$$L^n := \max_{i=1, \dots, m-1} |\mathbb{F}_i^n|, \quad (15)$$

then the following discrete energy inequality holds:

$$\begin{aligned} & \frac{\rho}{2\Delta t} (\|\mathbf{u}_h^{n+1}\|_{0,\Omega}^2 - \|\mathbf{u}_h^n\|_{0,\Omega}^2) + \mu \|\nabla \mathbf{u}_h^{n+1}\|_{0,\Omega}^2 \\ & + \frac{\kappa}{2\Delta t} \left(\left\| \frac{\partial \mathbf{X}_h^{n+1}}{\partial s} \right\|_{0,D}^2 - \left\| \frac{\partial \mathbf{X}_h^n}{\partial s} \right\|_{0,D}^2 \right) \leq \frac{C\kappa \Delta t}{2 h_x} L^n \|\nabla \mathbf{u}_h^{n+1}\|_{0,\Omega}^2. \end{aligned} \quad (16)$$

Theorem 2 gives a quantitative estimate of the artificial energy introduced into the system by the FE/BE numerical discretization.

For the problem to remain stable, i.e. with bounded energy, it is evident that some care has to be taken on the choice of the time step size, the fluid mesh size and the immersed boundary mesh size. The following lemma summarizes the CFL condition needed in order to maintain the property of decreasing total energy:

Lemma 3. *If there exists a positive K_0 such that for each $n = 0, \dots, N-1$*

$$\mu - \frac{C\kappa \Delta t}{2 h_x} L^n \geq K_0 > 0, \quad (17)$$

then the following discrete energy inequality holds:

$$\begin{aligned} & \frac{\rho}{2} \|\mathbf{u}^n\|_{0,\Omega}^2 + \Delta t \sum_{k=1}^n K_0 \|\nabla \mathbf{u}^k\|_{0,\Omega}^2 + \frac{\kappa}{2} \left\| \frac{\partial \mathbf{X}^n}{\partial s} \right\|_{0,D}^2 \\ & \leq \frac{\rho}{2} \|\mathbf{u}_0\|_{0,\Omega}^2 + \frac{\kappa}{2} \left\| \frac{\partial \mathbf{X}^0}{\partial s} \right\|_{0,D}^2. \end{aligned} \quad (18)$$

5 Numerical Results

To verify numerically the results stated in Theorem 2 and Lemma 3 we set up the extremely simple test problem of a balloon at rest inflated and immersed in the same fluid, which translate in our numerical framework in a circle with radius $R \leq .5$ immersed in the middle of the square domain $[0, 1]^2$ (we used $R = .4$), with null initial velocity \mathbf{u} and initial parametric representation given by

$$\mathbf{X}(s, t) = \begin{pmatrix} R \cos(s/R) + .5 \\ R \sin(s/R) + .5 \end{pmatrix} \quad s \in [0, 2\pi R]. \quad (19)$$

We are interesting in showing the dependency of the stability on the CFL parameter given by

$$\eta^n = \frac{\kappa \Delta t}{h_x} L^n. \quad (20)$$

In Figure 1 we plotted the evolution of the normalized total energy of the system and of the η^n parameter during time for different values of κ and Δt . All the computations we performed show how the η^n parameter (20) is able to capture the instabilities as soon as they arise.

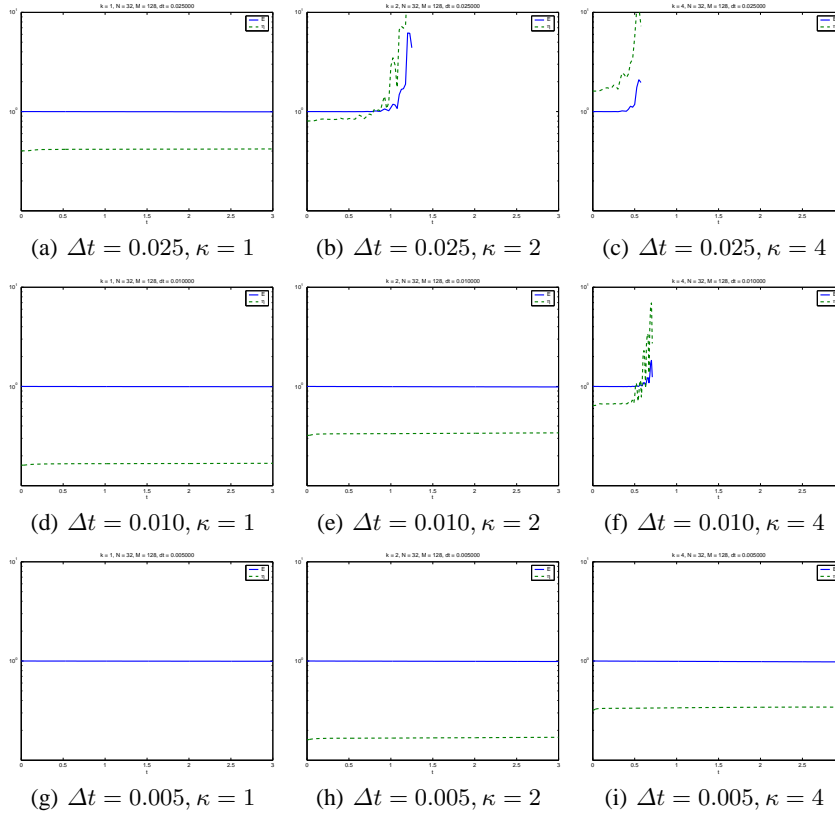


Fig. 1. Time versus normalized total energy and η , for $\kappa = 1$ (left), $\kappa = 2$ (middle) and $\kappa = 4$ (right) with $h_x = 1/32$ and $h_s = 1/128$.

It is evident that when the η^n parameter gets too close to a threshold, which here seems to be near .8, then the energy (which here is supposed to remain constant) explodes. The simulation in these cases stops without reaching the final time $t = 3$, because the immersed boundary starts oscillating too heavily and it ends outside the computational domain.

The program used to compute these examples has been written in C++ with the support of *deal.II* libraries(see [1] for a technical reference).

6 Conclusions

We recalled the formulation of the finite element immersed boundary method as found in [2, 3]. Choosing the correct strategy for the approximation of the delta distribution has been one of the major challenges for the developers of the original IB method (see, for example, [5]). In this paper we presented a numerical stability analysis of the finite element IB method, which in particular does not depend on a regularization of the Dirac delta distribution. The unconditional stability for the fully implicit time stepping technique (referred to as the BE/BE scheme) and a CFL condition for the semi-implicit time stepping technique (FE/BE) were presented. Previous work in this direction was carried on in [7, 6], by analyzing the vibrational modes of immersed fibers and their influence on the time-stepping technique. Our approach follows a somewhat different path, by asking the numerical method to satisfy physical conditions like the conservation of the total energy of the system. The numerical experiments we performed show good agreement between the theoretical results and the instability that sometimes arise during the IB problem computations.

References

1. Wolfgang Bangerth, Ralf Hartmann, and Guido Kanschat. *deal.II Differential Equations Analysis Library, Technical Reference*. <http://www.dealii.org>.
2. D. Boffi and L. Gastaldi. A finite element approach for the immersed boundary method. *Comput. & Structures*, 81(8-11):491–501, 2003. In honor of Klaus-Jürgen Bathe.
3. D. Boffi, L. Gastaldi, and L. Heltai. A finite element approach to the immersed boundary method. In Scotland Saxe-Coburg Publications, Stirling, editor, *Progress in Engineering Computational Technology, B.H.V. Topping and C.A. Mota Soares Eds.*, volume Chapt.12, pages 271–298, 2004.
4. F. Brezzi and M. Fortin. *Mixed and hybrid finite element methods*, volume 15 of *Springer Series in Computational Mathematics*. Springer-Verlag, New York, 1991.
5. C. S. Peskin. The immersed boundary method. In *Acta Numerica, 2002*. Cambridge University Press, 2002.
6. John M. Stockie and Brian R. Wetton. Analysis of stiffness in the immersed boundary method and implications for time-stepping schemes. *J. Comput. Phys.*, 154(1):41–64, 1999.
7. John M. Stockie and Brian T. R. Wetton. Stability analysis for the immersed fiber problem. *SIAM J. Appl. Math.*, 55(6):1577–1591, 1995.
8. C. Tu and C. S. Peskin. Stability and instability in the computation of flows with moving immersed boundaries: a comparison of three methods. *SIAM J. Sci. Stat. Comput.*, 13(6):1361–1376, 1992.