On a Model of Granular Flow: Global smooth solution, global BV solution and slow erosion limit

Wen Shen
Penn State University, U.S.A. Email: shen_w@math.psu.edu

We consider a 2×2 system of balance laws describing the flow of granular matter (e.g., an avalanche):

$$\begin{cases} h_t &= \operatorname{div}(h\nabla u) - (1 - |\nabla u|)h\,, \\ u_t &= \big(1 - |\nabla u|\big)h\,. \end{cases} \tag{1}$$

This model was proposed in [2]. Here h is the thickness of the moving layer, while u is the height of the standing layer. Assuming that $p \doteq u_x > 0$, differentiating w.r.t. x we obtain the system of balance laws

$$\begin{cases} h_t - (hp)_x &= (p-1)h\,, \\ p_t + \big((p-1)h\big)_x &= 0\,. \end{cases} \tag{2}$$

This system (2) is weakly linearly degenerate at the point $(h,p) = (0,1)$. The first characteristic field is genuinely nonlinear away from the line $p = 1$ and the second field is genuinely nonlinear away from the line $h = 0$. With suitable conditions on the initial data, a result on the global existence of smooth solutions of the system (2) was recently proved in [3]. In particular, we assumed that $u_x(0,x) \equiv 1$ for x outside a compact interval.

For more general initial data, the solution will develop discontinuities in finite time, due to nonlinearity of the flux functions. In [1] we prove the global existence of BV solutions, for a class of initial data with bounded but possibly large total variation. The main assumptions on the initial data $(\bar h, \bar p)$ are:

$$\bar h \in \mathbf{L}^1 \cap BV\,, \qquad \bar h \in \mathbf{L}^1 \cap BV\,, \qquad \bar p(x) \geq p_0 > 0\,.$$

Moreover, we assume that $\|h\|_{\mathbf{L}^\infty} \leq \delta$ for some δ sufficiently small.

Furthermore, we also study the "slow erosion (or deposition) limit" in [1]. We show that, if the thickness of the moving layer remains small, then the profile of the standing layer depends only on the total mass of the avalanche flowing downhill, not on the time-law describing at which rate the material slides down.

More precisely, we consider an initial-boundary value problem for (2), with prescribed flux at the boundary:

$$\begin{aligned} h(0,x) = \bar h(x), \quad p(0,x) = \bar p(x) \qquad & \text{for } x \leq 0, \\ p(t,0)h(t,0) = F(t) \qquad & \text{for } t \geq 0\,. \end{aligned}$$

We proved the global existence of BV solution for this initial-boundary value problem, provided that $\|\bar h\|_{\mathbf{L}^\infty} \leq \delta$ and $\|F\|_{\mathbf{L}^\infty} \leq \delta$ for sufficiently small δ. Furthermore, in the limit as the thickness of the moving layer tends to zero, the slope of the mountain provides the unique entropy solution to the scalar integro-differential conservation law

$$p_\mu + \left(\frac{p-1}{p} \cdot \exp \int_x^0 \frac{p(y)-1}{p(y)}\,dy\right)_x = 0\,.$$

Here μ is a rescaled time variable, implicitly defined by $\mu(t) = \int_0^t F(\tau)\,d\tau$, accounting for the total mass flowing downstream across the point $x = 0$.

References

[1] Debora Amadori and Wen Shen, Global Existence and the Slow Erosion/Deposition Limit in a Model of Granular Flow, Preprint 2008.

[2] K. P. Hadeler and C. Kuttler, Dynamical models for granular matter. *Granular Matter*, **2** (1999), 9–18.

[3] Wen Shen, On the shape of avalanches, *J. Math. Anal. Appl.*, **339** (2008), 828-838.