Phase-field Models for Transition Phenomena in Materials with Hysteresis

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Many real-world phenomena exhibit hysteresis:

- elasto-plastic solids,
- shape memory alloys,
- ferromagnetic materials,
- ferroelectric materials,
- Schmitt triggers (circuits)
- cell division (biology),
- activation in lymphoid cells (immunology)
- export performance (economics)
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Many mathematical models for hysteresis have been proposed, most of which are devoted to ferromagnetic bodies:

- Preisach (1935),
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**THE FOCUS of this mini-course is on:**

- **non-isothermal** hysteresis modeling compatible with thermodynamics;
- the **relation** between hysteresis and phase-transition, when the temperature $\theta$ is varying.

In particular, we stress the **DIFFERENCE** between

- **hysteretic phase-transitions**, typical in **first order** models (for instance, when melting temperature and freezing temperature do not agree);
- **structural hysteresis**, typical in **second order** models, when the transition occurs between a non-hysteretic and an hysteretic regime (or phase) at a given temperature (for instance, when paramagnetic material becomes ferromagnetic below the Curie temperature).
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In this framework we discuss two simple one-dimensional models:

1. **in Shape Memory Alloys**: A phase-field model (first order) which describes both temperature-induced and stress-induced phase transitions from austenitic to (oriented) martensitic phases (*pseudo-elastic regime*).

2. **in Ferromagnetics**: A phase-field model (second order) which describes both temperature-induced and $H$-induced phase transitions from paramagnetic (non-hysteretic) to ferromagnetic (hysteretic) regimes.

Finally, we suggest to apply the latter to describe the transition from pseudo-elastic to elasto-plastic regimes in Shape Memory Alloys. In addition to the former, this will produce a complete SMA model, that is a model which works at all temperatures.
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First order transitions without hysteresis.

Figure: The water P–T diagram
First order transitions without hysteresis.

Figure: Liquid-vapor transitions without hysteresis at constant temperature $\theta^*$ (red) and at constant pressure $p^*$ (blue).
Phase-field models for 1st order transitions

- Pressure-induced liquid/vapor transition at constant temperature \( \theta^* \).

**Figure:** The isothermal Amagat-Andrews diagram
Phase-field models for 1st order transitions

\[ P_{eq} = p^* \]

**Figure:** Maxwell construction of the Amagat-Andrews diagram
Temperature-induced liquid/vapor transition at constant pressure $p^*$. 

**Figure:** Isobaric diagram in the energy-temperature plane
Phase-field models for 1st order transitions without hysteresis.
LIQUID – VAPOR
Phase variable: $\varphi = 0$ vapor, $\varphi = 1$ liquid
for instance: $\varphi = c_\ell$ (liquid concentration) or $\varphi = \gamma(c_\ell)$
Ginzburg-Landau equation:

$$\rho \dot{\varphi} = -\kappa \delta \varphi \psi$$

$$\delta \varphi \psi = \rho \partial \varphi \hat{\psi} - \nabla \cdot (\rho \partial \nabla \varphi \hat{\psi})$$

Thermodynamic potential: Gibbs free energy $\psi$
for instance:

$$\psi(p, \theta, \varphi, |\nabla \varphi|) =$$

$$c_p \theta (1 - \ln \theta) + k \theta \left[ \ln \frac{p}{k} - 1 \right] + \lambda(\theta)g(\varphi)[u(p, \theta) - g(\varphi)] + \frac{1}{2} \nu |\nabla \varphi|^2$$

where $c_p, k, \lambda, \nu > 0 \ g = \gamma^{-1}$ and $u(p, \theta) \approx \frac{p}{p^*} - \beta(\theta/\theta^* - 1)$

Berti A., –, Morro, Mathematical modeling of phase transition and separation in fluids, DCDS-B (to appear)
Phase-field models for 1st order transitions

\[ \psi \]
\[ \theta < \theta^* \quad a) \]
\[ \theta = \theta^* \quad b) \]
\[ \theta > \theta^* \quad c) \]

Figure: **Isobaric** \((p = p^*\), above) and **isothermal** \((\theta = \theta^*\), below) free energy minima: in red the stable states.

\[ \psi \]
\[ p > p^* \quad a) \]
\[ p = p^* \quad b) \]
\[ p < p^* \quad c) \]
Phase-field models for 1st order transitions

- $u(p, \theta) > 1 \rightarrow \text{liquid}$ is stable,
- $u(p, \theta) < 1 \rightarrow \text{vapor}$ is stable,
- $u(p, \theta) = 1 \rightarrow \text{vapor-pressure curve (red)}$
Phase-field models for 1st order transitions with hysteresis.

AUSTENITE – MARTENSITE

Phase variable: \( \varphi = 0 \) austenite, \( \varphi = 1 \) martensite \( M^+ \)
\( \varphi = -1 \) martensite \( M^- \)

for instance: \( \varphi = \varepsilon_p/\varepsilon_t \) (normalized plastic strain) or
\( \varphi = \gamma(\varepsilon_p/\varepsilon_t) \) that is \( \varepsilon_p = \varepsilon_t g(\varphi) \), \( g = \gamma^{-1} \)

Ginzburg-Landau equation:

\[
\rho \dot{\varphi} = -\kappa \delta_{\varphi} \psi \\
\delta_{\varphi} \psi = \rho \partial_{\varphi} \psi - \nabla \cdot (\rho \partial_{\nabla \varphi} \hat{\psi})
\]

WARNING 1: the G-L equation is NOT rate-independent

Thermodynamic potential: Gibbs free energy \( \psi = \Psi + \frac{1}{2} |\nabla \varphi|^2 \)

WARNING 2: there are infinitely many (sub)potentials \( \Psi \)
1. The Gibbs free energy construction

Duhem’s rate-independent models are considered as starting point of the Gibbs free energy construction:

\[
\frac{d\sigma}{d\varepsilon} = \mathcal{F}(\sigma, \varepsilon, \text{sgn} \dot{\varepsilon}), \quad \text{sgn} \, P = \begin{cases} 
+1 & \text{if } P > 0, \\
0 & \text{if } P = 0, \\
-1 & \text{if } P < 0.
\end{cases}
\]

\(\sigma\) - stress, \(\varepsilon\) - total strain.

The role of skeleton curve description is emphasized.

The minimum (Gibbs) free energy representation \(\Psi_m\) is obtained by computing the maximum recoverable work.

\(\Psi_m\) is uniquely determined by the skeleton curve.
2 – The SMA bilinear model

\[
\frac{d\sigma}{d\varepsilon} = \begin{cases} 
\alpha & \text{if } (\varepsilon, \sigma) \in \Sigma_1 \cup \Sigma_2 \text{ or } (\varepsilon, \sigma) \in \Xi_1 \text{ and } \text{sgn} \dot{\varepsilon} = 1 \text{ or } (\varepsilon, \sigma) \in \Xi_2 \text{ and } \text{sgn} \dot{\varepsilon} = -1 \\
0 & \text{otherwise},
\end{cases}
\]

where \( y = y(\theta) \) and

\[
\Sigma_1 = \left\{ (\varepsilon, \sigma) : \sigma = \alpha \varepsilon, \ 0 \leq \varepsilon < \frac{y}{\alpha} \right\}
\]

\[
\Sigma_2 = \left\{ (\varepsilon, \sigma) : \sigma = \alpha \varepsilon - \frac{\alpha + \kappa}{\kappa} h, \ \varepsilon > \frac{y}{\alpha} + \frac{h}{\kappa} \right\}
\]

\[
\Xi_1 = \left\{ (\varepsilon, \sigma) : y - h \leq \sigma < y, \frac{\sigma}{\alpha} < \varepsilon < -\frac{\sigma}{\kappa} + \frac{\alpha + \kappa}{\alpha \kappa} y \right\} \cup \left( \frac{y}{\alpha} + \frac{h}{\kappa}, y - h \right)
\]

\[
\Xi_2 = \left\{ (\varepsilon, \sigma) : y - h < \sigma \leq y, -\frac{\sigma}{\kappa} + \frac{\alpha + \kappa}{\alpha \kappa} y < \varepsilon < \frac{\sigma}{\alpha} + \frac{\alpha + \kappa}{\alpha \kappa} h \right\} \cup \left( \frac{y}{\alpha}, y \right)
\]
3 – The graph of the bilinear model

\[ \varepsilon = \varepsilon_e + \varepsilon_p = \frac{1}{\alpha} \sigma + \varepsilon_t g(\varphi) \]

\( \varepsilon_p = 0 \) in the pure austenite phase \( \varphi = 0 \),
\( \varepsilon_p = \pm \varepsilon_t \) in the pure martensite phases \( \varphi = \pm 1 \).

Figure: Major and minor hysteresis loops (in red the skeleton curve).
First order transitions with hysteresis at constant stress

Figure: The \( (\theta, \sigma) \)-diagram: temperature-induced transitions.
First order transitions with hysteresis at constant stress

1 – Temperature-induced austenite/martensite transition

Shape memory alloys (pseudo-elastic regime: $\theta > \theta_A$)

Figure: The $A \rightarrow M^+$ ($\circ$) and $M^+ \rightarrow A$ ($\Box$) temperature-induced transitions at $\sigma = \sigma^* > 0$. 
Phase-field models for 1st order transitions

First order isothermal transitions with hysteresis: $\theta = \theta^*$

Figure: The $(\theta, \sigma)$-diagram: stress-induced transitions.

 pseudo-elastic
Phase-field models for 1st order transitions

First order isothermal transitions with hysteresis

1 – Stress-induced austenite/martensite transition

Bilinear model (pseudo-elastic regime: $\theta = \theta^* > \theta^*_A$)

Figure: Stable (solid) and unstable (dashed) equilibrium branches.
Phase-field models for 1st order transitions

First order isothermal transitions with hysteresis
2 – Stress-induced austenite/martensite transition

Devonshire model (pseudo-elastic regime: $\theta = \theta^* > \theta^*_A$)

\[ \sigma = y(\theta) - h - y(\theta) + h \]

Figure: Stable (solid) and unstable (dashed) equilibrium branches.
Construction of the minimum free energy in the bilinear case

Figure: Work recovered to reach the origin starting from $\xi_2$. 

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Isothermal 1st order transitions with hysteresis

Construction of the minimum free energy in the bilinear case

\[ \sigma > 0 \]

\[ \psi(\varphi, \sigma, \theta, \nabla \varphi) = \psi(\varphi, \sigma, \theta) + \frac{1}{2} \nu |\nabla \varphi|^2 \]

\[ \psi_m(\varepsilon_p, \sigma, \theta^*) = -\frac{1}{2\alpha} \sigma^2 + [y(\theta^*) - \sigma] \varepsilon_p - \frac{h}{2\varepsilon_t} \varepsilon_p^2. \]

After replacing \[ \varepsilon_p = \varepsilon_t g(\varphi) \]

we have

\[ \psi_m(\varphi, \sigma, \theta^*) = -\frac{1}{2\alpha} \sigma^2 + [y(\theta^*) - \sigma] \varepsilon_t g(\varphi) - \frac{h}{2\varepsilon_t} g^2(\varphi) \]

where \[ \theta = \theta^* > \theta^*_A \]

is a fixed temperature and

\[ g(0) = 0, \quad g(1) = 1, \quad g'(\varphi) \geq 0 \]

For instance, \[ g(\varphi) = \varphi^2(3 - 2\varphi) \] so that \( \psi \) turns out to be a polynomial of the sixth order.
Isothermal 1st order transitions with hysteresis

Different choices of $g$:

\[ g_1(\varphi) = \varphi^2(3 - 2\varphi), \quad g_2 = \varphi^2(2 - \varphi^2), \quad g_3 = \frac{1}{2}(1 - \cos \pi \varphi) \]

Figure: The graphs of $g_1$ (dashed), $g_2$ (blue) and $g_3$ (red) on $(0, 1)$.
**The minimum free energy**

Usually, $\Psi$ is a sixth-order (at least) polynomial: $g = g_1(\varphi)$. Why? $\Psi$ has to exhibit 3 minima and 2 maxima when $y - h < \sigma < y$.

![Graph showing the minimum free energy with critical points and regions](image)
The Ginzburg-Landau equation

\[ \dot{\varphi} = -\kappa \left[ \partial_\varphi \Psi_m(\varphi, \sigma) - \nu \Delta \varphi \right] \]
\[ = -\kappa \varepsilon_t \left[ y(\theta) - |\sigma| - hg(\varphi) \right] g'(\varphi) + \kappa \nu \Delta \varphi, \]

Since \( \dot{\varepsilon} = \dot{\varepsilon}_p + \dot{\varepsilon}_e = \varepsilon_t g'(\varphi) \dot{\varphi} + \frac{1}{\alpha} \dot{\sigma}, \)
it follows (neglecting diffusion \( \nu = 0) \)

\[ \dot{\varepsilon} = -\kappa \varepsilon_t^2 \left[ y(\theta) - |\sigma| - hg(\varphi) \right] [g'(\varphi)]^2 + \frac{1}{\alpha} \dot{\sigma} \]

Rate-type (not rate-independent) constitutive equation.

\[ \dot{\sigma} = \alpha \dot{\varepsilon} + \kappa \alpha \varepsilon_t^2 \left[ y(\theta) - |\sigma| - hg(\varphi) \right] [g'(\varphi)]^2. \]


Cycle simulation (isothermal: $y = y(\theta^*)$)

Assuming that $\sigma(t) = A \sin \omega t$ we obtain the system

$$
\begin{align*}
\dot{\varepsilon} &= -\kappa \varepsilon_t^2 \left[ y - |\sigma| - h g(\varphi) \right] \left[ g'(\varphi) \right]^2 + \frac{1}{\alpha} A \omega \cos \omega t \\
\dot{\sigma} &= A \omega \cos \omega t.
\end{align*}
$$

Choosing $g(\varphi) = \frac{1}{2}(1 - \cos \pi \varphi)$ and taking into account that

$$g(\varphi) = (\alpha \varepsilon - \sigma)/\alpha \varepsilon_t, \quad \left[ g'(\varphi) \right]^2 = \pi^2 g(\varphi)[1 - g(\varphi)]$$

we get

$$\left[ g'(\varphi) \right]^2 = \frac{\pi^2}{\alpha^2 \varepsilon_t^2} (\alpha \varepsilon - \sigma) \left[ \alpha (\varepsilon_t - \varepsilon) + \sigma \right]$$

and then

$$
\begin{align*}
\dot{\varepsilon} &= -\kappa \pi^2 \left[ y - |\sigma| - h \frac{\alpha \varepsilon - \sigma}{\alpha \varepsilon_t} \right] \left( \alpha \varepsilon - \sigma \right) \left[ \alpha (\varepsilon_t - \varepsilon) + \sigma \right] + \frac{1}{\alpha} A \omega \cos \omega t \\
\dot{\sigma} &= A \omega \cos \omega t.
\end{align*}
$$
**G-L model for 1st order transitions with hysteresis**

**Cycle simulation** (isothermal: $y = y(\theta^*)$)

The major loop is obtained by solving this (closed) system with IC $\varepsilon = \sigma = 0$

**Figure:** Numerical simulation (solid), theoretical shape (dashed).
**G-L model for 1st order transitions with hysteresis**

**Cycle simulation** *(isothermal: \( y = y(\theta^*) \))*

**Figure:** The minor loop: numerical simulation (solid), theoretical shape (dashed) starting from the origin \( \varepsilon = \sigma = 0 \)
G-L model for 1st order transitions with hysteresis

**Cycle simulation** (isothermal: $y = y(\theta^*)$)

The major loop with kinematical hardening: $\sigma = \sigma' + \beta \varepsilon$

**Figure:** Numerical simulation (solid), theoretical shape (dashed) starting from the origin $\varepsilon = \sigma = 0$
Conclusions

- Starting from a (simple) Duhem’s model accounting for hysteresis loops, we construct the corresponding G-L model by way of the computation of $\Psi_m$ (minimum free energy).
- $\Psi_m$ depends on the skeleton curve of the original model, only.
- The original Duhem’s model is able to account for minor loops, the final G-L model is not.
- The original Duhem’s model is rate-independent, the final G-L model is not.
- The final G-L model is almost rate-independent, provided that $\kappa$ is properly chosen according to the operating frequencies $\omega$.
- The final G-L model may be very easily coupled with PDEs (mass, momentum and energy balances) in order to rule the evolution of all variables; the original Duhem’s model may not.
Second order transitions without hysteresis.

SUPERCONDUCTIVITY

\[ \theta \]

\[ N \]

\[ H \]

\[ u(H, \theta) > 1 \]

\[ u(H, \theta) < 1 \]

\[ \theta_c \]

\[ O \]

\[ S \]

\[ H_0 \]

\[ N \]

\[ -H_0 \]

Figure: The \((\theta, H)\)-diagram in superconductivity: \(N\) = normal state, \(S\) = superconducting state. \(u(H, \theta) = 1\): the separation curve (blue),
Phase-field models for superconductivity.

Phase variable: \( \phi \in \mathbb{C} \), \( |\phi| = 0 \) normal state, 
\( |\phi| = 1 \) superconducting state 

\( |\phi|^2 = f_s \) (relative density of superconducting electrons)

Ginzburg-Landau equation (real \( \phi \in [-1, 1] \)):

\[
\Psi' = -\kappa \delta_\phi \Psi \\
\delta_\phi \Psi = \rho \partial_\phi \hat{\psi} - \nabla \cdot (\rho \partial_\nabla \phi \hat{\psi})
\]

Thermodynamic potential: Gibbs free energy \( \psi \)

for instance:

\[
\psi(H, \theta, \phi, |\nabla \phi|) =
\]

\[
c\theta(1 - \ln \theta) + \frac{1}{4} a |\phi|^2 \{ |\phi|^2 + 2[u(H, \theta) - 1] \} + \frac{1}{2} \nu |\nabla \phi|^2.
\]

where \( c, a, \nu > 0 \) and \( u(H, \theta) = H^2 / H_0^2 + \theta / \theta_c \)

Phase-field models for superconductivity

Isomagnetic transitions: \( H = H_* \in [0, H_0) \)

Free energy minima: in red the stable states.

\[
\theta_* = \theta_c \left[ 1 - \frac{H_*^2}{H_0^2} \right]
\]
Second order transitions with hysteresis.

FERROMAGNETISM

$\mathcal{N} = $ non-hysteretic states, $\mathcal{H} = $ hysteretic states

Figure: The $(\theta, H)$-diagram and the graphs of the skeleton curves when $\theta = \theta_1 > \theta_c$ (at the center) and $\theta = \theta_2 < \theta_c$ (on the right).


**Ferromagnetic materials:**

Paramagnetic regime: $\theta > \theta_c$

- $H = \text{external (applied) magnetic field}$
- $M = \text{magnetization}$
- $M_s = \text{maximum magnetization (saturation)}$

**Figure:** The paramagnetic regime: a) bilinear and b) Langevin.

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Phase-field Models in Materials with Hysteresis
Ferromagnetic materials:

1. Ferromagnetic regime: \( \theta < \theta_c \)
   - \( H = \) external (applied) magnetic field
   - \( H_c = \) coercive external magnetic field
   - \( M_r = \) residual magnetization

**Figure:** The major hysteresis loop: a) bilinear and b) Langevin.
**Ferromagnetic materials:**

2. **Ferromagnetic regime:** $\theta < \theta_c$
   - $\tilde{H} = H - \alpha M, \, \alpha > 0$ (internal magnetic field)
   - $\tilde{H}_c$ = coercive internal magnetic field

![Diagram](image_url)

**Figure:** The major hysteresis loop: a) bilinear and b) Langevin.

**Hint:** $\tilde{H}, \tilde{H}_c$ and $M$ play the role of $\sigma, y$ and $\varepsilon_p$ in SMA.
1. Duhem’s starting models

Bilinear model

\[
\frac{dM}{dH} = \begin{cases} 
\chi(\theta) & \text{if } M = f_b(H), \ |M| < M_s, \text{ or } \\
M = f_b(H), \ |M| = M_s \text{ and } M \text{ sgn } \dot{H} < 0, \text{ or } \\
M \neq f_b(H) \text{ and } [f_b(H) - M] \text{ sgn } \dot{H} > 0, \\
0 & \text{otherwise.}
\end{cases}
\]

Figure: Major loop and hysteresis path (arrowhead) in the bilinear model (skeleton curve \( f = f_b \) is red).
1. Duhem’s starting models

**Coleman & Hodgdon model** (with a Langevin skeleton $f_L$)

\[
\frac{dM}{dH} = \alpha [f_L(H) - M] \text{sgn} \dot{H} + g(H).
\]

**Figure:** The Coleman-Hodgdon model (skeleton curve $f = f_L$ in red and fatness $g = g_L$ in blue (dashed).
2. Temperature-induced transitions

\textit{a) The role of the skeleton curve}

- The slope of the skeleton curve at $H = 0$ depends on the temperature:

\[ \chi|_{H=0} = \chi_s(\theta) = \frac{\chi_0(\theta)}{1 + \gamma \chi_0(\theta)}, \quad \chi_0(\theta) = \frac{C}{\theta}, \quad \gamma = \alpha - \frac{\theta_c}{C}, \]

- In the limit of high temperatures $\chi_s(\theta) \approx C/(\theta - \theta_c)$ (Curie-Weiss law)
- There is a critical temperature, $\theta_c$, and a critical slope, $\chi_s(\theta_c) = 1/\alpha$, at which transition to hysteresis occurs.
- Soft materials: $\lim_{\theta \to 0} \chi_s(\theta) = 1/\gamma > 0$,
- Hard materials: $\lim_{\theta \to 0} \chi_s(\theta) = 1/\gamma < 0$, 
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2. Temperature-induced transitions

b) The role of the skeleton curve: the bilinear case

\[ \theta > \theta_c \]

\[ M \]

\[ M_s \]

\[ -M_s \]

\[ H \]

\[ \theta = \theta_c \]

\[ M \]

\[ M_s \]

\[ -M_s \]

\[ H \]

\[ \theta < \theta_c \]

\[ M \]

\[ M_s \]

\[ -M_s \]

\[ H \]

Figure: The bilinear-model transition: the critical slope (in red).
2. Temperature-induced transitions

c) The role of the skeleton curve: the Langevin case

\[ \theta \geq \theta_c \]

\[ \theta < \theta_c \]

Figure: The Langevin-model transition: the slope \( \chi|_{H=0} \) (dotted red).
2. Temperature-induced transitions

d) The role of the skeleton curve: soft and hard ferromagnetics

\[ M_s - M \quad \text{soft} \]
\[ M_s - M \quad \text{hard} \]

\[ \chi_s \approx \frac{1}{\gamma} > 0 \]
\[ \chi_s \approx \frac{1}{\gamma} < 0 \]

Figure: The bilinear-model transition: the skeleton slope when \( \theta \approx 0 \) (in red).

SOFT: small magnetic product \( M_r H_c \) \quad HARD: large magnetic product \( M_r H_c \)
3 - The internal magnetic field

- **Internal magnetic field** $\tilde{H}$ (Brown, 1963):
  \[
  \tilde{H} = H - \mathbf{A} \mathbf{M},
  \]

  $\mathbf{A}$ is a positive-definite tensor which depends on the shape and the anisotropy of the material.

- Projection along a fixed direction (eigenvector of $\mathbf{A}$)
  \[
  \tilde{H} = H - \alpha \mathbf{M}, \quad \alpha > 0
  \]

- Paramagnetic relation (Coey, 2009)
  \[
  M = f(\tilde{H}, \theta) = f(H - \alpha M, \theta), \quad f(0, \cdot) = 0.
  \]

  and
  \[
  \chi(H, \theta) = \partial_{\tilde{H}} f(\tilde{H}, \theta)
  \]
3 - The internal magnetic field

- **Internal magnetic field** $\tilde{H}$ (Brown, 1963):
  \[ \tilde{H} = H - A M, \]

  $A$ is a positive-definite tensor which depends on the **shape** and the **anisotropy** of the material.

- Projection along a fixed direction (eigenvector of $A$)
  \[ \tilde{H} = H - \alpha M, \quad \alpha > 0 \]

- **Paramagnetic relation** (Coey, 2009)
  \[ M = f(\tilde{H}, \theta) = f(H - \alpha M, \theta), \quad f(0, \cdot) = 0. \]

  and
  \[ \chi(H, \theta) = \partial_{\tilde{H}} f(\tilde{H}, \theta) \]
3 - The internal magnetic field

- **Internal magnetic field** \( \tilde{H} \) (Brown, 1963):
  \[
  \tilde{H} = H - AM,
  \]
  \( A \) is a positive-definite tensor which depends on the shape and the anisotropy of the material.

- Projection along a fixed direction (eigenvector of \( A \))
  \[
  \tilde{H} = H - \alpha M, \quad \alpha > 0
  \]

- **Paramagnetic relation** (Coey, 2009)
  \[
  M = f(\tilde{H}, \theta) = f(H - \alpha M, \theta), \quad f(0, \cdot) = 0.
  \]
  and
  \[
  \chi(H, \theta) = \partial_{\tilde{H}} f(\tilde{H}, \theta)
  \]
By reversing the paramagnetic relation we have

\[ H = f^{-1}(M, \theta) + \alpha M \]

and then

\[ M = \tilde{f}(H, \theta), \quad \tilde{\chi}(H, \theta) = \partial_H \tilde{f}(H, \theta) \]

\[ \tilde{\chi}|_{H=0} = \frac{\chi_s(\theta)}{1 - \alpha \chi_s(\theta)}, \quad \alpha = \frac{1}{\chi_s(\theta_c)} \]

- The critical slope \( \tilde{\chi}|_{H=0} \) at \( \theta = \theta_c \) becomes a vertical line.
- In the limit of high temperatures \( \tilde{\chi}(0, \theta) \approx \frac{C}{\theta - \theta_c} \)
  (Curie-Weiss law)
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- The critical slope \( \tilde{\chi}|_{H=0} \) at \( \theta = \theta_c \) becomes a vertical line.
- In the limit of high temperatures \( \tilde{\chi}(0, \theta) \approx C/(\theta - \theta_c) \) (Curie-Weiss law)
Temperature-induced transitions in the $\tilde{H} - M$ plane

The bilinear skeleton curve referred to the internal field at different temperatures.

C. Giorgi  HYSTRI, Milan, may 27-31 2013  Phase-field Models in Materials with Hysteresis
Temperature-induced transitions in the $\tilde{H} - M$ plane

The Langevin skeleton curve referred to the internal field.
The Ginzburg-Landau model

The choice of the phase variable:

\[ m = \frac{M}{M_s}, \quad |m| \leq 1 \]

from the general theory (Fabrizio,–, Morro, 2009)

\[ \dot{m} = -\kappa \delta_m \psi_G = -\kappa [\partial_m \psi_G - \nabla \cdot \partial_{\nabla m} \psi_G], \]

\[ \psi_G - \text{Gibbs free energy density (}\rho = 1), \]

\[ \psi_G = \psi - \tilde{H}B = V(M, \theta) + \frac{1}{2} \nu |\nabla M|^2 - \frac{1}{2} \mu_0 \tilde{H}^2 - \mu_0 \tilde{H}M, \]

\[ (1) \quad \Rightarrow \quad \dot{M} = -\kappa M_s^2 \left[ \partial_M V - \mu_0 \tilde{H} - \nabla \cdot (\nu \nabla M) \right]. \]
Assuming uniform fields ($\nabla M = 0$)

$$
\dot{M} = -\hat{\kappa} M_s^2 \left[ \partial_M V - \mu_0 \tilde{H} \right] = -\hat{\kappa} \partial_M \Phi, \quad \hat{\kappa} = \kappa M_s^2,
$$

where

$$
\Phi(\tilde{H}, M, \theta) = V(M, \theta) - \mu_0 \tilde{H} M, \quad \text{(Lagrangian density)}.
$$

**Problem**: how to take the expressions of $V$ and $\Phi$?

- $V$ can be uniquely determined from the skeleton curve:

$$
dV = \mu_0 \tilde{H} dM = \mu_0 f^{-1}(M, \theta) dM
$$

- $\Phi$ can be uniquely identified (to within a function of $\tilde{H}$) as the minimum Gibbs free energy

**Remark**: $\Phi(0, M, \theta) = V(M, \theta)$
Convex potentials $V$: $\theta > \theta_c$, $\tilde{H} = 0$

Figure: The graph of $V_b(\cdot, \theta)$ and $V_L(\cdot, \theta)$ when $\theta > \theta_c$. 
Non-convex potentials $V$: $0 < \theta < \theta_c$, $\tilde{H} = 0$

Figure: The graph of $V_b(\cdot, \theta)$ and $V_L(\cdot, \theta)$ when $0 < \theta < \theta_c$. 
Convex potentials $\Phi$: $\theta > \theta_c$, $\tilde{H} > 0$

Figure 5.1. The graph of $\Phi_b(\tilde{H}, \cdot, \theta)$ and $\Phi_L(\tilde{H}, \cdot, \theta)$ at $\theta > \theta_c$ when $\tilde{H} = 2\tilde{H}^*$ (solid), $\tilde{H} = \tilde{H}^*/2$ (dashed), $\tilde{H} = 0$ (red dashed).
Non-convex potentials $\Phi$: $\theta < \theta_c$, $\tilde{H} > 0$

Figure 5.2. The graph of $\Phi_b(\tilde{H}, \cdot, \theta)$ and $\Phi_L(\tilde{H}, \cdot, \theta)$ at $\theta < \theta_c$ when $\tilde{H} = 2\tilde{H}^*$ (solid), $\tilde{H} = \tilde{H}^*/2$ (dashed), $\tilde{H} = 0$ (red dashed).
The Langevine case:
temperature-induced transition in the $\tilde{H} - M$ plane

\begin{itemize}
\item[(a)] $M > M_s$ \quad $\theta > \theta_c$
\item[(b)] $M = M_s$ \quad $\theta = \theta_c$
\item[(c)] $M < M_s$ \quad $0 < \theta < \theta_c$
\end{itemize}
In the Langevin case from (2) we obtain

\[ \dot{M} = -\hat{\kappa} \mu_0 [\mathcal{L}(M, \theta) - \tilde{H}] , \]

where

\[ \mathcal{L}(M, \theta) = H_c(\theta) \mathbb{L}^{-1}(M/M_s) - \beta M, \]

\[ H_c(\theta) = \frac{M_s \theta}{3C}, \quad \beta = \frac{\theta_c}{C} > 0 \]

\[ \mathbb{L}(u) = \coth u - 1/u \quad \text{(Langevin function)} \]

Assuming \( \tilde{H}(t) = A \sin \omega t \), we obtain the closed system,

\[
\begin{aligned}
\dot{M} &= -\hat{\kappa} \mu_0 \left[ H_c(\theta) \mathbb{L}^{-1}(M/M_s) - (\tilde{H} + \beta M) \right] \\
\dot{H} &= A \omega \cos \omega t .
\end{aligned}
\]
Numerical simulation of the Langevine model in the \( \tilde{H} - M \) plane:

HARD MATERIALS

\[
\begin{align*}
\text{Ferromagnetic responses under a cyclic process in } \tilde{H} \text{ at different temperatures: } \theta_1 > \theta_c \text{ on the left, } \theta_2 = \theta_* < \theta_c \text{ in the center, } \theta_3 < \theta_* \text{ on the right. Simulations (solid) and skeleton curves (dashed).}
\end{align*}
\]
**Numerical simulation of the Langevine model in the H – M plane:**

**HARD MATERIALS**

Ferromagnetic responses under a cyclic process in $H = H_{\text{ex}}$ at different temperatures: $\theta_1 > \theta_c$ on the left, $\theta_2 = \theta_* < \theta_c$ in the center, $\theta_3 < \theta_*$ on the right. Simulations (solid) and skeleton curves (dashed).
The bilinear case:

temperature-induced transition in the \( \tilde{H} - M \) plane

\[
\begin{align*}
\chi \gamma > \frac{1}{\alpha} & \quad \theta_* < \theta < \theta_c \\
\chi \gamma < \frac{1}{\gamma} & \quad 0 < \theta < \theta_*
\end{align*}
\]

Figure: Hard ferromagnetic materials: \( \gamma < 0 \). The graphs in the \((H, M)\)-plane (\( \chi \gamma \) is the skeleton slope and \( \theta^* = C|\gamma| \)).
In the bilinear case, the Lagrangian function reads (see Figg.5.1.a and 5.2.a)

\[ \Phi_b(M, \tilde{H}, \theta) = \mu_0 I_{(-1,1)}(M/M_s) + \frac{\mu_0}{2\chi_{\beta}(\theta)} M^2 - \mu_0 M \tilde{H}. \]

and assuming \( M = M_s m \), from (2) we obtain

\[ \dot{M} = -\kappa \mu_0 \left[ \partial I_{(-1,1)} + \frac{M}{\chi_{\beta}(\theta)} - \tilde{H} \right], \]

where

\[ I_{(-1,1)} = \text{Indicator function of } (-1,1) \]
\[ \partial I_{(-1,1)} = \text{subdifferential of } I_{(-1,1)} \]
\[ \chi_{\beta}(\theta) = \frac{C}{\theta - \theta_c}, \]
SMOOTHING of $\Phi_b$

If we assume $M = M_s \Gamma(m)$, $\Gamma(m) = \frac{1}{2}(1 - \cos \pi m)$, then $l_{(-1,1)}$ can be removed from the potential,

$$\Phi_{b,\Gamma} = \frac{\mu_0}{2\chi_\beta(\theta)} M_s^2 \Gamma^2(m) - \mu_0 M_s \tilde{H} \Gamma(m).$$

Assuming $\tilde{H}(t) = A \sin \omega t$, from (2) we obtain the closed system,

$$\begin{cases} 
\dot{M} = -\frac{\pi^2 \hat{\kappa} \mu_0}{4} (M_s^2 - M^2) \left( \frac{M}{\chi_\beta(\theta)} - \tilde{H} \right) \\
\dot{H} = A\omega \cos \omega t.
\end{cases}$$

whose solutions can be easily computed by numerical methods.
Numerical simulation of the bilinear model: $\theta > \theta_c$

Paramagnetic response under a cyclic process in the $H - M$ plane at $\theta > \theta_c$. Simulations (solid) and theoretical curves (dashed).
Numerical simulation of the bilinear model: $\theta < \theta_c$

(soft material)

Ferromagnetic response under a cyclic process in the $H - M$ plane at $0 < \theta < \theta_c$. Simulations (solid) and theoretical curves (dashed).
Numerical simulation of the bilinear model: $\theta < \theta_c$

(hard material)

Ferromagnetic response under a cyclic process in the $H - M$ plane at $0 < \theta < \theta_\ast < \theta_c$. Simulations (solid) and theoretical curves (dashed).
2nd order transitions with hysteresis

Conclusions

- Starting from some **Duhem’s models** accounting for hysteresis loops (bilinear, Coleman-Hodgdon), we construct the corresponding **G-L model** by way of the computation of $\Phi$ (Lagrangian function)

\[ \Phi(M, H, \theta) = V(M, \theta) - \mu_0 HM \]

- $\Phi(M, H, \theta) = V(M, \theta) - \mu_0 HM$ depends on the **skeleton curve** of the original model.

- The original Duhem’s model is able to account for **minor loops**, the final G-L model is **not**.

- The original Duhem’s model is **rate-independent**, the final G-L model is **not**.

- The final G-L model is **almost rate-independent**, provided that $\kappa$ is properly chosen according to the operating frequencies $\omega$.

- The final G-L model may be **very easily coupled** with PDEs (mass, momentum and energy balances) in order to rule the evolution of all variables; the original Duhem’s model may **not**.