Interplay of viscosity and dry friction in rate-independent evolutions with nonconvex energies

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joint work (in progress) with
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WIAS, Berlin, 22.04.2009

Rate-independent evolutions in the applications

1. quasistatic propagation of fracture **[Bourdin, Cagnetti, Chambolle, Dal Maso, Francfort, Giacomini, Knees, Larsen, Lazzaroni, Marigo, Mielke, Negri, Ortner, Ponsiglione, Toader, Zanini ......]**
2. quasistatic phase transformations in shape memory alloys (SMA) **[Auricchio, Levitas, Mainik, Mielke, Theil, Roubíček, Stefanelli....]**
3. elastoplasticity: linearized & finite-strain **[Dal Maso, DeSimone, Fiaschi, Francfort, Mora, Morini, Mielke, Mainik, Roubíček...]**
4. damage **[Francfort, Garroni, Larsen, Mielke, Roubíček, Thomas...]**
5. delamination **[Kočvara, Mielke, Roubíček, Scardia, Zanini...]**
6. ferromagnetism, ferroelectricity, superconductivity **[Mielke, Schmid, Timofte...]**
7. shape evolution of debonding membranes **[Bucur, Buttazzo..]**

In these applications

- Typical **energies** are **nonsmooth & nonconvex**
- **Ambient spaces** may lack **a natural linear structure** (e.g. in crack propagation)

Energetic formulation for rate-independent evolutions

Weak formulations ("derivative-free")

Based on

- **energetic balance** (energy identity)
- **stability** conditions
- possibly enforcing **irreversibility**

Energetic formulation for rate-independent evolutions

Abstract approach by Mielke

- Ambient space $\mathcal{U}$ topological space
- Dissipation: $\mathcal{D} : \mathcal{U} \times \mathcal{U} \to [0, +\infty]$ pseudo-distance
- Energy: $\mathcal{E} : (0, T) \times \mathcal{U} \to (-\infty, +\infty]$

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Energetic formulation [Mielke-Theil'99,'04], [Mielke-Theil-Levitas'02], [Mainik-Mielke'05]

Global energetic solutions $u : [0, T] \to \mathcal{U}$: **global stability condition** & **energy balance**

$$\mathcal{E}(t, u(t)) - \mathcal{E}(t, z) \leq \mathcal{D}(u(t), z) \qquad \forall\, z \in \mathcal{U},$$

$$\mathcal{E}(t, u(t)) + \mathrm{Diss}_{\mathcal{D}}(u, [0, t]) = \mathcal{E}(0, u(0)) + \int_0^t \partial_t \mathcal{E}(r, u(r))\, \mathrm{d}r\,.$$

$\mathrm{Diss}_{\mathcal{D}}$ being the global dissipation functional associated with $\mathcal{D}$

The convex case

In **[Mielke-Theil'04]**: if

- ambient space $\mathcal{U}$ is a **reflexive** Banach space $\mathcal{B}$
- $\mathcal{E}(t,\cdot)$ (uniformly) **convex** & **smooth**
- $\mathcal{D}$ induced by $\Psi_1 : \mathcal{B} \to [0,+\infty)$ **convex & 1-positively homogeneous** ($\Psi_1 \sim \|\cdot\|$)

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then

- $u \in \mathrm{AC}([0,T];\mathcal{B})$ (even Lipschitz in time)
- the energetic formulation is **equivalent** to the **doubly nonlinear** equation

$$\partial\Psi_1(u'(t)) + \partial\mathcal{E}(t,u(t)) \ni 0 \quad \text{in } \mathcal{B}' \quad t \in (0,T)$$

(subdifferential formulation)

with $\partial\mathcal{E}(t,\cdot)$ **convex subdifferential** of $\mathcal{E}(t,\cdot)$ w.r.t. u

$$\xi \in \partial\mathcal{E}(t,u) \quad \Leftrightarrow \quad \mathcal{E}(t,w) - \mathcal{E}(t,u) \geq \langle \xi, w-u\rangle \text{ for all } w \in \mathcal{B}$$

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The nonconvex case: towards local stability

If $\mathcal{E}(t,\cdot)$ is nonconvex

- Ψ_1, 1-homogeneous, has a **linear growth** at ∞ $\rightsquigarrow$

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Problem with the Global Energetic formulation

Global stability forces global energetic solutions to **jump too early** and overcome **too large energy barriers** to avoid energy losses

Bad Vs. Good jumps

The simplest nonconvex case

$$\begin{cases} B = \mathbb{R}\,, \qquad \Psi_1(v) = |v| \;\; \forall\, v \in \mathbb{R} \\ \mathcal{E}(t,u) = \mathcal{W}(u) - \ell(t)u \qquad \forall\,(t,u) \in [0,T] \times \mathbb{R} \end{cases}$$

- $\mathcal{W}$ **double well** potential
- $\ell \in \mathrm{C}^1([0,T]) \sim$ external loading

$$\mathrm{Sign}(u'(t)) + \mathcal{W}'(u(t)) \ni \ell(t), \quad t \in (0,T)$$

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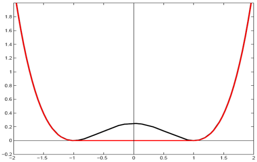
Convexification $\mathcal{W}^{**}$ of $\mathcal{W}$



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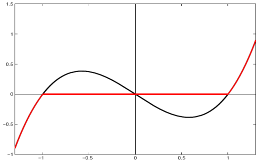
Global solutions are given by $u(t) = (D\mathcal{W}^{**})^{-1}\,(\ell(t) - 1)$: jumping too early!



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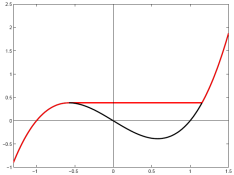
We aim to model the "right" hysteresis dynamics



The vanishing viscosity approach

Aim:

Formulation for rate-independent problems

- **modelling** only **"natural" jumps** (due to $u \in \mathrm{BV}(0, T; \mathcal{B})$),
- leading to **solutions jumping later** than global energetic solutions

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The vanishing viscosity method:

select rate-independent evolutions arising in the limit of viscous regularizations. In:

- plasticity with softening: **[Dal Maso-DeSimone-Mora-Morini'08]**, nonconvex elastoplasticity: **[Fiaschi'09]**.
- crack propagation: **[Toader-Zanini'06], [Knees-Mielke-Zanini'07, '08], [Cagnetti'07]**.
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The **vanishing viscosity** approach leads to formulations oriented towards **local stability** **[Dal Maso-Toader'02], [Negri-Ortner'07], (crack propagation), [Stefanelli'08]..**

The vanishing viscosity approach

Problem

In the vanishing viscosity limit:

- local stability
- energy inequality

may not be enough for controlling jumps. ¿ Which further conditions better describe them?

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An abstract approach

Approximation by vanishing viscosity of

- doubly nonlinear rate-independent equations (**subdifferential formulation**)
- in an abstract Banach setting, with general **nonconvex** energy functionals

extending **previous analysis by Efendiev-Mielke [Efendiev-Mielke'06]**

The technique by Efendiev-Mielke

A reparametrization technique

- We are modelling systems with two **time scales**:
  - a scale **intrinsic** to the system, **fast time scale**
  - the **slow time scale** of the **external loading** $\sim \partial_t \mathcal{E}$ (**dominating scale**) $\rightsquigarrow$ viscous dissipation is negligible!

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- Jumps in the vanishing viscosity limit correspond to **viscous transitions** between stable states
- To capture the viscous transition path: **NOT SHRINK** jumps at a point, look at curves with their **arc length parametrization**
- Asymptotic analysis of (reparametrized) trajectories in an extended phase space

Nonconvex energies

[Mielke-R.-Savaré, in progress] In infinite dimensions:

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Approximate

$$\partial\Psi(u'(t)) + \partial\mathcal{E}(t, u(t)) \ni 0 \quad \text{in } \mathcal{B}' \quad t \in (0, T), \tag{DNE}$$

Ψ **1-positively homogeneous**

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- $u \mapsto \mathcal{E}(t, u)$ is nonconvex $\Rightarrow$ choice of a **suitable subdifferential notion $\boldsymbol{\partial}\mathcal{E}$**
- **jumps in the rate-independent evolution** $\Rightarrow$ to be modelled via vanishing viscosity

Outline

- existence results for "viscous" doubly nonlinear equations with nonconvex energies
- two ideas on the vanishing viscosity analysis

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- existence results for "viscous" doubly nonlinear equations with nonconvex energies
- two ideas on the vanishing viscosity analysis
- parametrized rate-independent evolutions:
  - **local stability**
  - **jumps** $\leftrightarrow$ **viscous transitions** between metastable states

Which subdifferential notion? Heuristics

$$\begin{cases} \partial\Psi_\varepsilon(u'(t)) + \boldsymbol{\partial}\mathcal{E}(t,u(t)) \ni 0 \quad \text{in } \mathcal{B}' \quad t \in (0,T), \\ u(0) = u_0 \end{cases} \tag{DNE}$$

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Ψ_ε **with superlinear growth**

Step 1: approximation by time discretization

- Fix time step $\tau > 0 \quad \rightsquigarrow \quad$ partition $0 < \mathrm{t}_1 < \ldots \mathrm{t}_n = n\tau < \ldots < \mathrm{t}_N = T$ of $(0,T)$
- **Discrete solutions:** the recursive minimization problem

$$\text{Find } U_\tau^0, U_\tau^1, \ldots, U_\tau^N \in \mathcal{B}:$$

$$U_\tau^n \in \mathrm{Argmin}_{U \in \mathcal{B}} \{\tau\Psi_\varepsilon\left(\frac{U - U_\tau^{n-1}}{\tau}\right) + \mathcal{E}(t_n, U)\}, \quad U_\tau^0 := u_0$$

  has a solution if $\mathcal{E}(t,\cdot)$ is **coercive** (e.g., $\mathcal{E}(t,\cdot) + \|\cdot\|_{\mathcal{B}}^2$ has compact sublevels)
- **Discrete Euler-Lagrange equation:** $\partial_F\mathcal{E}(t,\cdot)$ the **Fréchet subdifferential** of $\mathcal{E}(t,\cdot)$

$$\partial\Psi_\varepsilon\left(\frac{U_\tau^n - U_\tau^{n-1}}{\tau}\right) + \partial_F\mathcal{E}(t_n, U_\tau^n) \ni 0,$$

The Fréchet subdifferential

Idea: "localize" the convex subdifferential

The Fréchet subdifferential

Idea: "localize" the convex subdifferential

The convex subdifferential

Given $u \in D(\mathcal{E}(t,\cdot))$,

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- $\partial_F \mathcal{E}(t,u) \equiv \partial \mathcal{E}(t,u)$ if $\mathcal{E}(t,\cdot)$ is convex
- $\partial_F \mathcal{E}(t,u)$ is **convex** for all $u \in D(\mathcal{E}(t,\cdot))$
- The map $u \rightrightarrows \partial_F \mathcal{E}(t,u)$ is **not strongly-weakly closed** in the sense of graphs!!!

Heuristics

Step 2: estimates on the approximate solutions

- **Approximate solutions**: interpolants on $(0,T)$ of the discrete solutions $\{U_\tau^k\}_{k=1}^n$:
  - $\{\overline{U}_\tau\}$ piecewise constant;
  - $\{\widehat{U}_\tau\}$ piecewise linear.
- from the discrete Euler-Lagrange equation $\rightsquigarrow$ **Approximate equation**:

$$\partial\Psi_\varepsilon\left(\widehat{U}_\tau'(t)\right) + \partial_F\mathcal{E}(t,\overline{U}_\tau(t)) \ni 0 \quad t\in(0,T)$$

- **A priori estimates + compactness** (strong for $\widehat{U}_\tau$, weak for $\widehat{U}_\tau'$)
- convergence to some limit curve $u \in W^{1,1}(0,T;\mathcal{B})$

BUT: you can't pass to the limit in

$$-\partial\Psi_\varepsilon\left(\widehat{U}_\tau'(t)\right) \ni \partial_F\mathcal{E}(t,\overline{U}_\tau(t)) \ni 0 \quad t\in(0,T)$$

because $u \rightrightarrows \partial_F\mathcal{E}(t,u)$ **in general** is **NOT strongly-weakly closed!!**

The limiting subdifferential

First idea:

Consider (a version of) the **strong-weak closure** of $\partial_F \mathcal{E}(t,u)$:

the limiting subdifferential $\partial\mathcal{E}(t,\cdot)$ [Mordukhovich'84]

given $u \in D(en(t,\cdot))$,

$$\xi \in \boldsymbol{\partial}\mathcal{E}(t,u) \quad \Leftrightarrow \quad \exists\,\{u_n\},\{\xi_n\} \subset \mathcal{B} : \begin{cases} \xi_n \in \partial_F \mathcal{E}(t,u_n) \ \forall\, n \in \mathbb{N}, \\ u_n \to u, \\ \xi_n \rightharpoonup \xi, \\ \sup_n \mathcal{E}(t,u_n) < +\infty \end{cases}$$

The **limiting subdifferential** is our notion of subdifferential!

Heuristics

Second idea:

- Instead of passing to the limit in the **pointwise** equation

$$-\partial\Psi_{\varepsilon}\left(\widehat{U}_{\tau}'(t)\right) \ni \partial_F \mathcal{E}(t, \overline{U}_{\tau}(t)) \ni 0 \quad t \in (0, T)$$

pass to the limit in the **approximate energy inequality** (**technical point!**)

$$\int_0^t \Psi_{\varepsilon}\left(\widehat{U}_{\tau}'(s)\right) \mathrm{d}s + \int_0^t \Psi_{\varepsilon}^*\left(-\partial_F \mathcal{E}(s, \overline{U}_{\tau}(s))\right) \mathrm{d}s + \mathcal{E}(t, \overline{U}_{\tau}(t)) \leq \mathcal{E}(0, u_0) + \int_0^t \partial_t \mathcal{E}(s, \overline{U}_{\tau}(s))\, \mathrm{d}s$$

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- By **LOWER-SEMICONTINUITY** obtain the **limit energy inequality**

$$\int_0^t \Psi_\varepsilon(u'(s))\,\mathrm{d}s + \int_0^t \Psi_\varepsilon^*\left(-\boldsymbol{\partial}\mathcal{E}(s, u(s))\right) \mathrm{d}s + \mathcal{E}(t, u(t)) \leq \mathcal{E}(0, u_0) + \int_0^t \partial_t \mathcal{E}(s, u(s))\,\mathrm{d}s$$

Heuristics

Third idea:

If $\boldsymbol{\partial}\mathcal{E}(t,\cdot)$ fulfils the **chain rule**

$$u \in W^{1,1}(0,T;\mathcal{B}) \;\Rightarrow\; \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,u(t)) = \langle \boldsymbol{\partial}\mathcal{E}(t,u(t)), u'(t)\rangle + \partial_t\mathcal{E}(t,u(t)) \text{ a.e. in } (0,T)$$

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then

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Third idea:

If $\boldsymbol{\partial}\mathcal{E}(t,\cdot)$ fulfils the **chain rule**

$$u \in W^{1,1}(0,T;\mathcal{B}) \;\Rightarrow\; \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,u(t)) = \langle \boldsymbol{\partial}\mathcal{E}(t,u(t)), u'(t)\rangle + \partial_t\mathcal{E}(t,u(t)) \text{ a.e. in } (0,T)$$

then

$$\begin{aligned}\int_0^t \Psi_\varepsilon(u'(s))\,\mathrm{d}s + \int_0^t \Psi_\varepsilon^*\left(-\boldsymbol{\partial}\mathcal{E}(s,u(s))\right)\mathrm{d}s + \mathcal{E}(t,u(t)) &\leq \mathcal{E}(0,u_0) + \int_0^t \partial_t\mathcal{E}(s,u(s))\,\mathrm{d}s \\ &= \mathcal{E}(t,u(t)) - \int_0^t \langle \boldsymbol{\partial}\mathcal{E}(s,u(s)), u'(s)\rangle\,\mathrm{d}s\end{aligned}$$

$$\int_0^t \Big(\Psi_\varepsilon(u'(s)) + \Psi_\varepsilon^*\left(-\boldsymbol{\partial}\mathcal{E}(s,u(s))\right) + \langle \boldsymbol{\partial}\mathcal{E}(s,u(s)), u'(s)\rangle\Big)\,\mathrm{d}s = 0$$

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whence

$$\partial\Psi_\varepsilon(u'(t)) + \boldsymbol{\partial}\mathcal{E}(t,u(t)) \ni 0 \quad \text{for a.a. } t \in (0,T).$$

and the **energy identity**

$$\int_0^t \Psi_\varepsilon(u'(s))\,\mathrm{d}s + \int_0^t \Psi_\varepsilon^*\left(-\boldsymbol{\partial}\mathcal{E}(s,u(s))\right)\mathrm{d}s + \mathcal{E}(t,u(t)) = \mathcal{E}(0,u_0) + \int_0^t \partial_t\mathcal{E}(s,u(s))\,\mathrm{d}s$$

An existence result

To sum it up

- Fréchet subdifferential $\partial_F \mathcal{E}(t,\cdot)$ naturally pops out from time-incremental minimization
- use with its strong-weak closure $\boldsymbol{\partial}\mathcal{E}(t,\cdot)$

An existence result

Existence theorem

Assume

- $u \mapsto \mathcal{E}(t, u)$ is **coercive**
- $t \mapsto \mathcal{E}(t, u)$ is **smooth** enough
- $\partial\mathcal{E}(t, \cdot)$ complies with the **chain rule**: for example if

$$\mathcal{E}(t, \cdot) = \mathcal{E}_{\text{convex}}(t, \cdot) + \mathcal{E}_{\text{concave}}(t, \cdot)$$

& the **convex part dominates concave part**

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Then, the approximate solutions converge to a curve $u \in W^{1,1}(0, T; \mathcal{B})$ which **solves** the Cauchy problem for

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Then, the approximate solutions converge to a curve $u \in W^{1,1}(0, T; \mathcal{B})$ which **solves** the Cauchy problem for

$$\partial\Psi_{\varepsilon}(u'(t)) + \boldsymbol{\partial}\mathcal{E}(t, u(t)) \ni 0 \quad \text{for a.a. } t \in (0, T).$$

Moreover, u fulfils the **energy identity** (due to the chain rule) for all $0 \le s \le t \le T$

$$\int_s^t \Psi_{\varepsilon}(u'(r))\,\mathrm{d}r + \int_s^t \Psi_{\varepsilon}^*(-\boldsymbol{\partial}\mathcal{E}(r, u(r)))\,\mathrm{d}r + \mathcal{E}(t, u(t)) = \mathcal{E}(0, u_0) + \int_s^t \partial_t\mathcal{E}(r, u(r))\,\mathrm{d}r\,.$$

Setting

Specialize to dissipation Ψ_ε on $\mathcal{B}$

$$\Psi_\varepsilon(\cdot) = \overbrace{\Psi_1(\cdot)}^{\text{rate-independent}} + \varepsilon\overbrace{\Psi_2(\cdot)}^{\text{viscous}}$$

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Further specialize to **dissipation given by norms!**

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Abstract rate-independent evolutions

- Dissipation:
  - B_2 reflexive
  - $B_2 \subset B_1 \equiv \mathcal{B}$
  - rate-independent dissipation: $\Psi_1(\cdot) = |\cdot|_1$
- Energy: $\mathcal{E} : (0, T) \times B_2 \to (-\infty, +\infty]$, $\mathcal{E}(t, \cdot)$ with **compact sublevels** in B_2, $\mathcal{E}(t, \cdot)$ **nonconvex**, smooth with respect to t

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$$\partial\Psi_1(u'(t)) + \boldsymbol{\partial}\mathcal{E}(t, u(t)) \ni 0 \quad \text{in } B_2',\ t \in (0, T)$$

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Interaction of L^1 & L^2 norms

- Dissipation:
  - $B_2 \sim L^2(\Omega)$
  - $B_1 \sim L^1(\Omega)$
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- Energy:

$$\mathcal{E}(t,u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \overbrace{\int_\Omega \mathcal{W}(u)}^{\text{double-well potential}} - \langle \ell(t), u\rangle$$

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$$\partial\Psi_1(u'(t)) + \boldsymbol{\partial}\mathcal{E}(t,u(t)) \ni 0 \quad \text{in } B_2',\ t\in(0,T)$$

$$\mathrm{Sign}(u_t) - \Delta u + \mathcal{W}'(u) = \ell(t) \quad \text{in } \Omega\times(0,T)$$

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- Dissipation:
  - $B_2 \sim L^2(\Omega)$
  - $B_1 \sim L^1(\Omega)$
  - rate-independent dissipation: $\Psi_1(\cdot) = |\cdot|_1$
  - **viscous approximation** $\Psi_\varepsilon(\cdot) = |\cdot|_1 + \frac{\varepsilon}{2}\|\cdot\|_2^2$
- Energy:

$$\mathcal{E}(t,u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \int_\Omega \mathcal{W}(u) - \langle \ell(t), u\rangle$$

$$\varepsilon\partial\Psi_2(u'(t)) + \partial\Psi_1(u'(t)) + \boldsymbol{\partial}\mathcal{E}(t,u(t)) \ni 0 \quad \text{in } B_2',\ t \in (0,T)$$

$$\varepsilon u_t + \mathrm{Sign}(u_t) - \Delta u + \mathcal{W}'(u) = \ell(t) \quad \text{in } \Omega \times (0,T)$$

Step 1: energy identity

Chain rule for $\partial\mathcal{E}$ + convex analysis: every solution of

$$\overbrace{\varepsilon\partial\Psi_2(u'(t)) + \partial\Psi_1(u'(t))}^{\partial\Psi_\varepsilon(u'(t))} + \boldsymbol{\partial}\mathcal{E}(t,u(t)) \ni 0 \text{ in } B_2', \ \ t\in(0,T)$$

fulfils **energy identity** $\forall\, 0 \le t_1 \le t_2 \le T$

$$\int_{t_1}^{t_2} \Psi_\varepsilon\left(\dot u(r)\right) \mathrm{d}r + \int_{t_1}^{t_2} \Psi_\varepsilon{}^*\left(-\boldsymbol{\partial}\mathcal{E}(r,u(r))\right) \mathrm{d}r + \mathcal{E}(t_2,u(t_2))$$
$$= \mathcal{E}(t_1,u(t_1)) + \int_{t_1}^{t_2} \partial_t\mathcal{E}(t,u(r))\,\mathrm{d}r$$

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$$= \mathcal{E}(t_1,u(t_1)) + \int_{t_1}^{t_2} \partial_t\mathcal{E}(t,u(r))\,\mathrm{d}r$$

where the conjugate $\Psi_\varepsilon{}^*$ is:

$$\Psi_\varepsilon{}^*\,(-\boldsymbol{\partial}\mathcal{E}) = \frac{1}{2\varepsilon}\min_{|z|_{1,*}\leq 1} \|-\boldsymbol{\partial}\mathcal{E} - z\|_{2,*}^2 = \frac{\mathbf{1}}{\mathbf{2\varepsilon}}\mathbf{d_2}\,(-\boldsymbol{\partial}\mathcal{E},\mathbf{K}^*)^{\mathbf{2}}$$

and K^* **unitary ball in** B_1^*.

Competition between viscous effects & rate-independent behaviour

$$\int_{t_1}^{t_2} |u'(r)|_1 \,\mathrm{d}r + \int_{t_1}^{t_2} \frac{\varepsilon}{2}\|u'(r)\|_2^2 + \frac{1}{2\varepsilon} d_2\left(-\boldsymbol{\partial}\mathcal{E}(r, u(r)), K^*\right)^2 \,\mathrm{d}r + \mathcal{E}(t_2, u(t_2))$$

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♣ Note that

$$d_2\left(-\boldsymbol{\partial}\mathcal{E}(r,u(r)), K^*\right) = 0 \quad \Leftrightarrow \quad |-\boldsymbol{\partial}\mathcal{E}(r,u(r))|_{1,*} \leq 1$$

LOCAL version of the stability condition:

$$|-\boldsymbol{\partial}\mathcal{E}(r,u(r))|_{1,*} \leq 1 \quad \text{vs.} \quad \frac{\mathcal{E}(t,u(t)) - \mathcal{E}(t,z)}{\mathcal{D}(u(t),z)} \leq 1 \;\forall\, z$$

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♣ Energy identity highlights the **competition between viscosity & rate-independence**

Step 2: rescaling

The technique by Efendiev-Mielke

- Jumps in the vanishing viscosity limit correspond to **viscous transitions** between stable states
- To capture the viscous transition path: **NOT SHRINK** jumps at a point, look at curves with their **arclength parametrization**
- Asymptotic analysis of (reparametrized) trajectories in an extended phase space

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- Asymptotic analysis of (reparametrized) trajectories in an extended phase space

We reparametrize

$$\int_{t_1}^{t_2} |u'(r)|_1 \,\mathrm{d}r + \int_{t_1}^{t_2} \frac{\varepsilon}{2}\|u'(r)\|_2^2 + \frac{1}{2\varepsilon} d_2\left(-\boldsymbol{\partial}\mathcal{E}(r,u(r)),K^*\right)^2 \,\mathrm{d}r + \mathcal{E}(t_2,u(t_2))$$
$$= \mathcal{E}(t_1,u(t_1)) + \int_{t_1}^{t_2} \partial_t \mathcal{E}(t,u(r))\,\mathrm{d}r$$

by the rescaling function:

$$s_\varepsilon(t) = t + \int_0^t \left(|u'(r)|_1 + \|u'(r)\|_2 \cdot d_2\left(-\boldsymbol{\partial}\mathcal{E}(r,u(r)),K^*\right)\right) \mathrm{d}r$$

"energy arclength"

Step 2: rescaling

$$
\begin{cases}
s_\varepsilon(t) := t + \int_0^t \left(|u'(r)|_1 + \|u'(r)\|_2 \cdot d_2\left(-\partial\mathcal{E}(r, u(r)), K^*\right)\right) \mathrm{d}r \\
\hat{t}_\varepsilon = s_\varepsilon^{-1}, \qquad \hat{u}_\varepsilon = u \circ \hat{t}_\varepsilon
\end{cases}
$$

Hence we perform the asymptotic analysis for the **"extended" trajectory** $\{(\hat{t}_\varepsilon, \hat{u}_\varepsilon)\}$

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Hence we perform the asymptotic analysis for the **"extended" trajectory** $\{(\hat{t}_\varepsilon, \hat{u}_\varepsilon)\}$

Rescaled energy identity

$$\int_{s_1}^{s_2} |\hat{u}_\varepsilon'|_1 + \int_{s_1}^{s_2} \frac{\varepsilon}{2\hat{t}_\varepsilon'} \|\hat{u}_\varepsilon'\|_2^2 + \frac{\hat{t}_\varepsilon'}{2\varepsilon} d_2\left(-\boldsymbol{\partial}\mathcal{E}(\hat{t}_\varepsilon, \hat{u}_\varepsilon), K^*\right)^2 + \mathcal{E}(\hat{t}_\varepsilon(s_2), \hat{u}_\varepsilon(s_2))$$
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$+$ **"normalization" condition**

$$\hat{t}_\varepsilon' + |\hat{u}_\varepsilon'|_1 + \|\hat{u}_\varepsilon'\|_2 \cdot d_2\left(-\boldsymbol{\partial}\mathcal{E}(\hat{t}_\varepsilon, \hat{u}_\varepsilon), K^*\right) \equiv 1$$

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$$\begin{aligned} &\int_{s_1}^{s_2} |\hat{u}_\varepsilon'|_1 + \int_{s_1}^{s_2} \frac{\varepsilon}{2\hat{t}_\varepsilon'} \|\hat{u}_\varepsilon'\|_2^2 + \frac{\hat{t}_\varepsilon'}{2\varepsilon} d_2\left(-\partial\mathcal{E}(\hat{t}_\varepsilon,\hat{u}_\varepsilon),K^*\right)^2 + \mathcal{E}(\hat{t}_\varepsilon(s_2),\hat{u}_\varepsilon(s_2)) \\ &\qquad = \mathcal{E}(\hat{t}_\varepsilon(s_1),\hat{u}_\varepsilon(s_1)) + \int_{s_1}^{s_2} \partial_t\mathcal{E}(\hat{t}_\varepsilon,\hat{u}_\varepsilon)\,\hat{t}_\varepsilon' \end{aligned}$$

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$\Rightarrow$ **A priori estimates, compactness** for $\{(\hat{t}_\varepsilon, \hat{u}_\varepsilon)\}$

The asymptotic analysis as $\varepsilon \downarrow 0$

Theorem [Mielke, R., Savaré'09]

- B_2 reflexive, $B_2 \subset B_1$
- Energy: $\mathcal{E} : (0, T) \times B_2 \to (-\infty, +\infty]$, with compact sublevels in B_2,
- $\mathcal{E}$ complies with **chain rule**

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Theorem [Mielke, R., Savaré'09]

Up to a subsequence, as $\varepsilon \searrow 0$ the trajectory $\{(\hat{t}_\varepsilon, \hat{u}_\varepsilon)\}$

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converges to $(\hat{t}, \hat{u}) \in \mathrm{C}^0_{\mathrm{Lip}}([0, \widehat{T}]; [0, T] \times B_1)$

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($|\hat{u}'|_1$ **metric derivative** of $\hat{u}$!), with

$$\mathcal{M}_0\left(\hat{t}', \|\hat{u}'\|_2, -\boldsymbol{\partial}\mathcal{E}(\hat{t}, \hat{u})\right) = \begin{cases} \|\hat{u}'\|_2 \cdot d_2\left(-\boldsymbol{\partial}\mathcal{E}(\hat{t}, \hat{u}), K^*\right) & \text{se } \hat{t}' = 0 \\ I_{\{0\}}\left(d_2\left(-\boldsymbol{\partial}\mathcal{E}(\hat{t}, \hat{u}), K^*\right)\right) & \text{se } \hat{t}' > 0 \end{cases}$$

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$$= \mathcal{E}(\hat{t}(s_1), \hat{u}(s_1)) + \int_{s_1}^{s_2} \partial_t \mathcal{E}(\hat{t}, \hat{u})\, \hat{t}'$$

($|\hat{u}'|_1$ **metric derivative** of $\hat{u}$!), with

$$\mathcal{M}_0\left(\hat{t}', \|\hat{u}'\|_2, -\boldsymbol{\partial}\mathcal{E}(\hat{t}, \hat{u})\right) = \begin{cases} \|\hat{u}'\|_2 \cdot d_2\left(-\boldsymbol{\partial}\mathcal{E}(\hat{t}, \hat{u}), K^*\right) & \text{se } \hat{t}' = 0 \\ I_{\{0\}}\left(d_2\left(-\boldsymbol{\partial}\mathcal{E}(\hat{t}, \hat{u}), K^*\right)\right) & \text{se } \hat{t}' > 0 \end{cases}$$

+ **"normalization condition"**

$$\hat{t}' + |\hat{u}'|_1 + \|\hat{u}'\|_2 \cdot d_2\left(-\boldsymbol{\partial}\mathcal{E}(\hat{t}, \hat{u}), K^*\right) \equiv 1$$

Two key ideas

Proof based on:

- reparametrization of trajectories
- energy identity: need for a notion of subdifferential for which the chain rule holds..

Remarks on the limit problem

$$(\hat{t},\hat{u}) \in \mathrm{C}^0_{\mathrm{Lip}}([0,\widehat{T}];[0,T]\times B_1), \qquad \hat{t}(0)=0, \quad \hat{t}(\widehat{T})=T$$

Normalization: $\hat{t}' + |\hat{u}'|_1 + \|\hat{u}'\|_2 \cdot d_2\left(-\boldsymbol{\partial}\mathcal{E}(\hat{t},\hat{u}), K^*\right) \equiv 1$

Energy id.:
$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\hat{t}(s),\hat{u}(s)) - \partial_t\mathcal{E}(\hat{(}s),\hat{u}(s))\,\hat{t}'(s)$$
$$= -\langle -\boldsymbol{\partial}\mathcal{E}(\hat{t}(s),\hat{u}(s)), \hat{u}'(s)\rangle$$
$$= -|\hat{u}'(s)|_1 - \|\hat{u}'(s)\|_2 \cdot d_2\left(-\boldsymbol{\partial}\mathcal{E}(\hat{t}(s),\hat{u}(s)), K^*\right) \quad \text{for a.a. } s \in (0,\widehat{T})$$

Constraint: $\mathcal{M}_0\left(\hat{t}', \|\hat{u}'\|_2, -\boldsymbol{\partial}\mathcal{E}(\hat{t},\hat{u})\right) \equiv \|\hat{u}'\|_2 \cdot d_2\left(-\boldsymbol{\partial}\mathcal{E}(\hat{t},\hat{u}), K^*\right)$

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$(\hat{t}, \hat{u}) \in \mathrm{C}^0_{\mathrm{Lip}}([0, \widehat{T}]; [0, T] \times B_1), \qquad \hat{t}(0) = 0, \quad \hat{t}(\widehat{T}) = T$

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$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\hat{t}(s), \hat{u}(s)) - \partial_t \mathcal{E}((\hat{s}), \hat{u}(s))\, \hat{t}'(s) \\
&= -\langle -\boldsymbol{\partial}\mathcal{E}(\hat{t}(s), \hat{u}(s)), \hat{u}'(s)\rangle \\
&= -|\hat{u}'(s)|_1 - \|\hat{u}'(s)\|_2 \cdot d_2\left(-\boldsymbol{\partial}\mathcal{E}(\hat{t}(s), \hat{u}(s)), K^*\right) \quad \text{for a.a. } s \in (0, \widehat{T})
\end{aligned}
$$

Three regimes:

$$
\begin{cases}
\hat{t}'(s) = 1 \ (\Leftrightarrow \hat{u}'(s) = 0\,) \\
\qquad \Rightarrow \ |-\boldsymbol{\partial}\mathcal{E}(\mathbf{r}, \mathbf{u}(\mathbf{r}))|_{1,*} \leq \mathbf{1} \\
\hat{t}'(s) \in (0,1) \ (\Leftrightarrow |\hat{u}'(s)|_1 \in (0,1)\,) \\
\qquad \Rightarrow \ |-\boldsymbol{\partial}\mathcal{E}(\hat{\mathbf{t}}(\mathbf{s}), \hat{\mathbf{u}}(\mathbf{s}))|_{1,*} = \mathbf{1} \\
\hat{t}'(s) = 0 \ (\Leftrightarrow \ |\hat{u}'(s)|_1 + \|\hat{u}'(s)\|_2 \cdot d_2\left(-\boldsymbol{\partial}\mathcal{E}(\hat{t}(s), \hat{u}(s)), K^*\right) = 1 \\
\qquad \Rightarrow \ |-\boldsymbol{\partial}\mathcal{E}(\hat{\mathbf{t}}(\mathbf{s}), \hat{\mathbf{u}}(\mathbf{s}))|_{1,*} \geq \mathbf{1}
\end{cases}
$$

Parametrized rate-independent evolutions

Definition

We call **parametrized rate-independent evolution** a pair

$$(\hat{t}, \hat{u}) \in \mathrm{AC}([0, \widehat{T}]; [0, T] \times B_1), \qquad \hat{t}(0) = 0, \quad \hat{t}(\widehat{T}) = T,$$

satisfying

Nondegeneracy: $\hat{t}'(s) + |\hat{u}'(s)|_1 > 0$ for a.a. $s \in (0, \widehat{T})$,

Energy id.:
$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\hat{t}(s), \hat{u}(s)) - \partial_t \mathcal{E}(\hat{(}s), \hat{u}(s))\, \hat{t}'(s)$$
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Three regimes:
$$\begin{cases} \mathbf{\hat{t}'(s) > 0 \ \Rightarrow\ |-\boldsymbol{\partial}\mathcal{E}(\hat{t}(s), \hat{u}(s))|_{1,*} \leq 1} \\ \mathbf{\hat{t}'(s)\, |\hat{u}'(s)|_1 > 0 \ \Rightarrow\ |-\boldsymbol{\partial}\mathcal{E}(\hat{t}(s), \hat{u}(s))|_{1,*} = 1} \\ \mathbf{\hat{t}'(s) = 0 \ \Rightarrow\ |-\boldsymbol{\partial}\mathcal{E}(\hat{t}(s), \hat{u}(s))|_{1,*} \geq 1} \end{cases}$$

Properties of parametrized rate-independent evolutions

Nondegeneracy: $\hat{t}'(s) + |\hat{u}'(s)|_1 > 0$ for a.a. $s \in (0, \widehat{T})$,

Energy id.:

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\hat{t}(s), \hat{u}(s)) - \partial_t \mathcal{E}(\hat{(}s), \hat{u}(s))\, \hat{t}'(s)$$

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- **Intrinsically rate-independent** notion: invariant for time-rescalings
- Arises in the **vanishing viscosity limit**
- Under conditions on B_1, **notion equivalent to the doubly nonlinear PDE formulation**
- **Stability w.r.t. the problem data**

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- Arises in the **vanishing viscosity limit**
- Under conditions on B_1, **notion equivalent to the doubly nonlinear PDE formulation**
- **Stability w.r.t. the problem data**
- **Three regimes reflect the different types of evolution of the system**

The three regimes

$$
\begin{cases}
\hat{t}'(s) > 0,\ |\hat{u}'(s)|_1 = 0 \Rightarrow & |-\partial\mathcal{E}(\hat{t}(s),\hat{u}(s))|_{1,*} \leq 1 \quad \textbf{(sticking)} \\
\hat{t}'(s)\,|\hat{u}'(s)|_1 > 0 \Rightarrow & |-\partial\mathcal{E}(\hat{t}(s),\hat{u}(s))|_{1,*} = 1 \quad \textbf{(sliding)} \\
\hat{t}'(s) = 0 \Rightarrow & |-\partial\mathcal{E}(\hat{t}(s),\hat{u}(s))|_{1,*} \geq 1 \quad \textbf{(viscous slip)}
\end{cases}
$$

The three regimes

$$\begin{cases} \hat{t}'(s) > 0, \ |\hat{u}'(s)|_1 = 0 \ \Rightarrow & |-\boldsymbol{\partial}\mathcal{E}(\hat{t}(s),\hat{u}(s))|_{1,*} \leq 1 \quad \textbf{(stationarity)} \\ \hat{t}'(s)\,|\hat{u}'(s)|_1 > 0 \ \Rightarrow & |-\boldsymbol{\partial}\mathcal{E}(\hat{t}(s),\hat{u}(s))|_{1,*} = 1 \quad \textbf{(rate-independent)} \\ \hat{t}'(s) = 0 \ \Rightarrow & |-\boldsymbol{\partial}\mathcal{E}(\hat{t}(s),\hat{u}(s))|_{1,*} \geq 1 \quad \textbf{(jump)} \end{cases}$$

Stationarity

- **local stability** condition

$$|-\boldsymbol{\partial}\mathcal{E}(\hat{t}(s_0),\hat{u}(s_0))|_{1,*} \leq 1 \ \text{ vs. } \ \frac{\mathcal{E}(s_0,u(s_0)) - \mathcal{E}(s_0,z)}{\mathcal{D}(u(s_0),z)} \leq 1 \ \forall\, z$$

- in a neighbourhood $I(s_0)$ there hold $\hat{u}(s) \equiv \hat{u}(s_0)$ and the **energy identity**

$$\mathcal{E}(\hat{t}(s_2),\hat{u}(s_0)) - \mathcal{E}(\hat{t}(s_1),\hat{u}(s_0)) = \int_{s_1}^{s_2} \partial_t\mathcal{E}(\hat{t}(s),\hat{u}(s_0))\,\hat{t}'(s)\,\mathrm{d}s \quad \forall\, s_1 \leq s_2 \in I(s_0)\,.$$

The three regimes

$$\begin{cases} \hat{t}'(s) > 0,\ |\hat{u}'(s)|_1 = 0 \ \Rightarrow & |-\boldsymbol{\partial}\mathcal{E}(\hat{t}(s), \hat{u}(s))|_{1,*} \leq 1 \quad \textbf{(stationarity)} \\ \hat{t}'(s)\,|\hat{u}'(s)|_1 > 0 \ \Rightarrow & |-\boldsymbol{\partial}\mathcal{E}(\hat{t}(s), \hat{u}(s))|_{1,*} = 1 \quad \textbf{(rate-independent)} \\ \hat{t}'(s) = 0 \ \Rightarrow & |-\boldsymbol{\partial}\mathcal{E}(\hat{t}(s), \hat{u}(s))|_{1,*} \geq 1 \quad \textbf{(jump)} \end{cases}$$

Rate-independent evolution

- **local stability** condition

$$|-\boldsymbol{\partial}\mathcal{E}(\hat{t}(s_0), \hat{u}(s_0))|_{1,*} = 1 \quad \sim \ -\boldsymbol{\partial}\mathcal{E}(\hat{t}(s_0), \hat{u}(s_0)) \in \overbrace{\partial\Psi_1(\hat{u}'(s_0))}^{\mathrm{Sign}(\hat{u}'(s_0))}$$

- in a neighbourhood $I(s_0)$ **energy identity** $\forall\, s_1 \leq s_2 \in I(s_0)$

$$\mathcal{E}(\hat{t}(s_2), \hat{u}(s_2)) - \mathcal{E}(\hat{t}(s_1), \hat{u}(s_1)) = \int_{s_1}^{s_2} \left(\partial_t \mathcal{E}(\hat{t}(s), \hat{u}(s))\, \hat{t}'(s) - |\hat{u}'(s)|_1\right) \mathrm{d}s\,.$$

The three regimes

$$\begin{cases} \hat{t}'(s) > 0, \ |\hat{u}'(s)|_1 = 0 \ \Rightarrow & |-\boldsymbol{\partial}\mathcal{E}(\hat{t}(s),\hat{u}(s))|_{1,*} \leq 1 \quad \textbf{(stationarity)} \\ \hat{t}'(s)\,|\hat{u}'(s)|_1 > 0 \ \Rightarrow & |-\boldsymbol{\partial}\mathcal{E}(\hat{t}(s),\hat{u}(s))|_{1,*} = 1 \quad \textbf{(rate-independent)} \\ \hat{t}'(s) = 0 \ \Rightarrow & |-\boldsymbol{\partial}\mathcal{E}(\hat{t}(s),\hat{u}(s))|_{1,*} \geq 1 \quad \textbf{(jump)} \end{cases}$$

Jump regime

- in a neighbourhood $I(s_0)$ there hold $\hat{t}(s) \equiv \hat{t}(s_0)$ and **energy identity** $\forall\, s_1 \leq s_2 \in I(s_0)$

$$\mathcal{E}(\hat{t}(s_0),\hat{u}(s_2)) - \mathcal{E}(\hat{t}(s_0),\hat{u}(s_1)) = \int_{s_1}^{s_2} \left(-|\hat{u}'(s)|_1 - \|\hat{u}'(s)\|_2 \cdot d_2\left(-\boldsymbol{\partial}\mathcal{E}(\hat{t}(s_0),\hat{u}(s)),K^*\right)\right)\,\mathrm{d}s\,.$$

- With a **rescaling** which depends on the

$$\textbf{viscous quantities} \quad \|\hat{u}'\|_2 \ \& \ d_2\left(-\boldsymbol{\partial}\mathcal{E}(\hat{t},\hat{u}),K^*\right)$$

  we pass from $\hat{u}(s)$ to $\tilde{u}(\sigma)$ solution of the **viscous doubly nonlinear equation**

$$\partial\Psi_1(\tilde{u}'(\sigma)) + \partial\Psi_2(\tilde{u}'(\sigma)) + \boldsymbol{\partial}\mathcal{E}(\hat{t}(s_0),u(\sigma)) \ni 0$$

Differential characterization

In the case

B_1 has the Radon Nikodým property

(hence we have **pointwise derivatives**!! But L^1 **is forbidden**) &

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then $(\hat{t}, \hat{u}) \in \mathrm{AC}([0, \widehat{T}]; [0,T] \times B_1)$ is a parametrized rate-independent evolution

if and only if

$$\exists\, \boldsymbol{\lambda} : (0, \widehat{T}) \to \mathbb{R}^+ \ : \text{for a.a. } s \in (0, \widehat{T})$$

$$\begin{cases} \partial\Psi_1(\hat{u}'(s)) + \boldsymbol{\lambda}(s)\partial\Psi_2(\hat{u}'(s)) + \boldsymbol{\partial}\mathrm{E}(\hat{u}(s)) \ni \boldsymbol{\ell}(\hat{t}(s)) \\ \boldsymbol{\lambda}(s)\hat{t}'(s) = 0 \end{cases}$$

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The "Lagrange multiplier" $\boldsymbol{\lambda}$ (and thus viscous dissipation)
is **activated by** $\hat{t}'(s) = 0$, i.e. at jumps

BV rate-independent evolutions

From virtual to real jumps

With a suitable rescaling, we pass

from $(\hat{t}, \hat{u})$ parametrized rate-independent evolution

$$\Downarrow \ \Uparrow$$

to u **BV rate-independent evolution**

in jump points it follows the trajectory of a **viscous doubly nonlinear equation**

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Approximation by time discretization

Existence of BV rate-independent evolutions: passing to the limit in the **discretization scheme**:

$$\begin{cases} U_\tau^n \in \mathrm{Argmin}_{U \in B_2} \left\{ |U - U_\tau^{n-1}|_1 + \frac{\varepsilon(\tau)}{\tau} \|U - U_\tau^{n-1}\|_2^2 + \mathcal{E}(t_n, U) \right\} . \\ \text{with } \ \varepsilon(\tau) \to 0 \ \& \ \frac{\varepsilon(\tau)}{\tau} \uparrow \infty \ \text{ as } \tau \downarrow 0 \end{cases}$$

Metric analysis popping out

$$\int_{s_1}^{s_2} |\hat{u}'|_1 + \int_{s_1}^{s_2} \mathcal{M}_0 \left(\hat{t}', \|\hat{u}'\|_2, -\boldsymbol{\partial}\mathcal{E}(\hat{t}, \hat{u})\right) + \mathcal{E}(\hat{t}(s_2), \hat{u}(s_2))$$
$$= \mathcal{E}(\hat{t}(s_1), \hat{u}(s_1)) + \int_{s_1}^{s_2} \partial_t \mathcal{E}(\hat{t}, \hat{u})\, \hat{t}'$$

BUT:

$B_1 \sim L^1$ does **not** have the **Radon-Nikodým property**

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$$
\begin{aligned}
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- Need to replace pointwise derivative $|\hat{u}'|_1$ by **metric derivative**
- Metric analysis, theory of **gradient flows** in metric spaces:
  - **De Giorgi, Marino, Saccon, Tosques, Degiovanni, Ambrosio** '80 $\sim$ '90 $\rightsquigarrow$ theory of **Curves of Maximal Slope** and **Minimizing Movements**

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$$
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- subdifferentials replaced by **slopes**, doubly nonlinear equations formulated in a metric setting **[R., Mielke, Savaré'08], [Mielke, R., Savaré'08]**

Future developments

- applications of this general method to **"concrete" problems in continuum mechanics**..
- **from two norms to two metrics**..