WED functionals for gradient flows in metric spaces: the convex case

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work in progress with
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Outline

- Gradient flows and WED functionals in Hilbert spaces
- Recaps on gradient flows in metric spaces
- The WED approach in metric spaces: difficulties
- A different viewpoint
- Convergence of the WED approximation in the convex case

Gradient flows in Hilbert spaces

Setup:

- $(\mathcal{H}, \|\cdot\|)$ separable Hilbert space
- $\varphi : \mathcal{H} \to (-\infty, +\infty]$ (proper), lower semicontinuous, convex functional
- $u_0 \in \mathrm{dom}(\varphi)$

Let's consider the Cauchy problem

$$\begin{cases} u'(t) + \partial\varphi(u(t)) \ni 0 \quad \text{in } \mathcal{H}, \quad t \in (0, T), \\ u(0) = u_0 \end{cases} \tag{P}$$

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with $\partial\varphi$ **convex-analysis subdifferential of φ:**

$$u \in D(\varphi),\ \xi \in \partial\varphi(u) \ \Leftrightarrow\ \varphi(w) - \varphi(u) \geq \langle \xi, w-u \rangle \quad \forall\, w \in \mathcal{H}$$

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"Classical" results on (P)

Existence, uniqueness, approximation of solutions: (**[Kōmura'67, Crandall-Pazy'69, Brézis'73]**)

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For fixed $\varepsilon > 0$, $I_\varepsilon : H^1(0,T;\mathcal{H}) \to (-\infty, \infty]$

$$I_\varepsilon(v) := \int_0^T \mathrm{e}^{-t/\varepsilon} \left(\frac{1}{2}\|v'(t)\|^2 + \frac{1}{\varepsilon}\varphi(v(t)) \right) \mathrm{d}t$$

is the **WED=Weighted Energy Dissipation** functional for (P)

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Every minimizer $u_\varepsilon \in H^1(0,T;\mathcal{H})$ satisfies

$$\begin{cases} -\varepsilon {u_\varepsilon}''(t) + u_\varepsilon'(t) + \partial\varphi(u_\varepsilon(t)) \ni 0 \text{ in } \mathcal{H}, \ t \in (0,T) & \text{(Eul.-Lagr. equation)}, \\ u_\varepsilon(0) = u_0 & \text{(initial condition)}, \\ u_\varepsilon'(T) = 0 & \text{(final condition)} \end{cases}$$

(this can be easily seen for **smooth** φ, calculating the first variation of I_ε)

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Problem: **for $\varepsilon \searrow 0$ (causal limit),** does (u_ε) converge to u, solution of (P)?
YES, see [Mielke-Stefanelli, ESAIM-COCV 2010].

Sketch of the proof (I)

$$\min_{v \in H^1(0,T;\mathcal{H}),\ v(0)=u_0} \int_0^T \mathrm{e}^{-t/\varepsilon} \left(\frac{1}{2}\|v'(t)\|^2 + \frac{1}{\varepsilon}\varphi(v(t)) \right) \mathrm{d}t \qquad \text{(MIN)}$$

Facts:

- φ convex & l.s.c. $\Rightarrow$ I_ε is (uniformly) convex on $H^1(0,T;\mathcal{H})$ & l.s.c.
- I_ε has a minimizer u_ε
- u_ε satisfies the Euler-Lagrange equation (can be checked via convex analysis)

$$\begin{cases} -\varepsilon u_\varepsilon''(t) + u_\varepsilon'(t) + \partial\varphi(u_\varepsilon(t)) \ni 0 \text{ in } \mathcal{H},\ t \in (0,T) & \text{(Eul.-Lagr. equation)}, \\ u_\varepsilon(0) = u_0 & \text{(initial condition)}, \\ u_\varepsilon'(T) = 0 & \text{(final condition)} \end{cases}$$

- Crucial estimate on (u_ε):

$$\exists\, C > 0 \ :\ \varepsilon\|u_\varepsilon''\|_{L^2(0,T;\mathcal{H})} + \varepsilon^{1/2}\|u_\varepsilon'\|_{L^\infty(0,T;\mathcal{H})} + \|u_\varepsilon'\|_{L^2(0,T;\mathcal{H})} + \|\partial\varphi(u_\varepsilon)\|_{L^2(0,T;\mathcal{H})} \le C$$

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under the further assumption $\varphi \in \mathrm{C}^{1,1}(\mathcal{H})$.

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- **Conclusion of the proof:**
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  - **passage to the limit in** (EUL) **as** $\varepsilon \searrow 0$

Outcome and applications of the results

The convergence of the WED minimizers (u_ε) to u (unique) solution of the gradient flow, is interesting

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2. **general theoretical frame**: approach to evolution problems via minimization of functionals on spaces of trajectories: **[Brézis-Ekeland principle 1976]**, **[Nayroles 1976]**; monography by **[Ghoussoub 2008]**, **[Stefanelli 2008–, Visintin 2008]**

The WED approach to metric gradient flows

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WED functional on $(0, +\infty)$

We address a **different problem**: convergence as $\varepsilon \searrow 0$ of

$$u_\varepsilon \in \mathrm{Argmin} \left\{ \int_0^{+\infty} \mathrm{e}^{-t/\varepsilon} \left(\frac{1}{2} \|v'(t)\|^2 + \frac{1}{\varepsilon} \varphi(v(t)) \right) \mathrm{d}t \right\}$$

$\rightsquigarrow$ links with the theory of **optimal control**, dynamic programming principle

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Further motivation

Obtain an **"alternative" proof of existence** of curves of maximal slope: the **WED functional has a "scalar character"**, hence it's suited to analysis in metric spaces, where derivatives are replaces by **"scalar surrogate"** quantities.

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So we get the **equivalent formulation**:

$$\frac{d}{dt}\varphi(u(t)) = -\frac{1}{2}\|u'(t)\|^2 - \frac{1}{2}\|\mathrm{D}\varphi(u(t))\|^2$$

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&\Leftrightarrow \|u'(t)\|^2 + \|\mathrm{D}\varphi(u(t))\|^2 + 2\frac{\mathrm{d}}{\mathrm{d}t}\varphi(u(t)) = 0
\end{aligned}
$$

So we get the **equivalent formulation**:

$$
\frac{d}{dt}\varphi(u(t)) = -\frac{1}{2}\|u'(t)\|^2 - \frac{1}{2}\|\mathrm{D}\varphi(u(t))\|^2
$$

involving **norms of derivatives,** rather than derivatives!!

Heuristics for the metric formulation of gradient flows

Suppose **for the time being** that $\varphi : \mathcal{H} \to (-\infty, +\infty]$ is **smooth**

$$
\begin{aligned}
u'(t) + \mathrm{D}\varphi(u(t)) = 0 &\Leftrightarrow \|u'(t) + \mathrm{D}\varphi(u(t))\|^2 = 0 \\
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involving **norms of derivatives,** rather than derivatives!!
Hence this formulation can be given in a metric context, using suitable **“surrogate notions” of (norms of) derivatives**.

"Derivatives" in metric spaces (I)

- **Ambient space:** (X, d) complete metric space

Metric derivative & geodesics

Given an **absolutely continuous curve** $u : [0, T] \to X$ ($u \in \mathrm{AC}([0, T]; X)$), its **metric derivative** is

$$|u'|(t) := \lim_{h \to 0} \frac{d(u(t), u(t+h))}{|h|} \quad \text{for a.a. } t \in (0, T),$$

$$\boxed{\|u'(t)\| \rightsquigarrow |u'|(t)}$$

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A curve $\gamma : [0, 1] \to X$ is a (constant-speed) **geodesic** if:

$$\frac{d(\gamma(s), \gamma(t))}{|t - s|} = d(\gamma(0), \gamma(1)) \doteq |\gamma'| \quad \forall\, s, t \in [0, 1].$$

“Derivatives” in metric spaces (II)

- **Ambient space:** (X, d) complete metric space

Local slope & Chain rule

- Given $\varphi : X \to (-\infty, +\infty]$ (proper), and $u \in D(\varphi)$, the **local slope** of φ at u is

$$|\partial\varphi|(u) := \limsup_{v \to u} \frac{(\varphi(u) - \varphi(v))^+}{d(u,v)} \quad u \in D(\varphi)$$

$$\boxed{\| - \mathrm{D}\varphi(u)\| \rightsquigarrow |\partial\varphi|(u)}$$

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- We say that $\varphi : X \to (-\infty, +\infty]$ satisfies the **chain rule** with respect to $|\partial\varphi|$ if $\forall\, v \in \mathrm{AC}([0, T]; D(\varphi))$, the map $t \mapsto \varphi(v(t))$ is **absolutely continuous** and

$$-\frac{\mathrm{d}}{\mathrm{d}t}\varphi(v(t)) \le |v'|(t)\, |\partial\varphi|(v(t)) \quad \text{for a.a. } t \in (0, T).$$

Metric formulation of gradient flows

Curves of Maximal Slope

An **absolutely continuous** curve $u : [0, T] \to X$ is a **Curve of Maximal Slope** for φ (w.r.t. the local slope) if

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\varphi(u(t)) = -\frac{1}{2}|u'|^2(t) - \frac{1}{2}|\partial\varphi|^2(u(t)) \quad \text{a.e. in } (0, T).} \tag{CMS}$$

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- In view of the chain rule $\boxed{-\frac{\mathrm{d}}{\mathrm{d}t}\varphi \circ v \leq |v'|\,|\partial\varphi(v)|}$, (CMS) is equivalent to the **inequality**

$$\frac{d}{dt}\varphi(u(t)) \leq -\frac{1}{2}|u'|^2(t) - \frac{1}{2}|\partial\varphi|^2(u(t)) \quad \text{a.e. in } (0, T).$$

Existence for Curves of Maximal Slope

Theorem [Ambrosio-Gigli-Savaré '05]

IF $\varphi : X \to (-\infty, +\infty]$

1. is proper, lower semicontinuous
2. "coercive" $\sim \varphi$ has compact sublevels
3. the local slope $u \mapsto |\partial\varphi|(u)$ is l.s.c.
4. φ satisfies the chain rule w.r.t. $|\partial\varphi|$

THEN for every $u_0 \in D(\phi)$ there **exists** a Curves of Maximal Slope u for φ, with $u(0) = u_0$.

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Remark

- A sufficient condition for properties (3) & (4) is

$$\boxed{\varphi \text{ is (geodesically-)convex}}$$

i.e. for all u_0, $u_1 \in D(\varphi)$ there exists a geodesic $\gamma : [0,1] \to X$ such that

$$\varphi(\gamma(\theta)) \leq (1-\theta)\varphi(\gamma(0)) + \theta\varphi(\gamma(1)) \quad \forall\, \theta \in [0,1].$$

WED functionals on $(0, +\infty)$

Basic assumptions on $\varphi : X \to (-\infty, +\infty]$

1. φ is proper, lower semicontinuous, positive (bdd from below)
2. φ has compact sublevels

♣ For fixed $\varepsilon > 0$, we consider $\mathcal{I}_\varepsilon : \mathrm{AC}(0, +\infty; X) \to (-\infty, \infty]$

$$\mathcal{I}_\varepsilon(v) := \int_0^{+\infty} \mathrm{e}^{-t/\varepsilon} \left(\frac{1}{2}|v'|^2(t) + \frac{1}{\varepsilon}\varphi(v(t)) \right) \mathrm{d}t$$

with $|v'|$ **metric derivative** of v.

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Fact (I)

The WED minimum problem

$$\min \{ \mathcal{I}_\varepsilon(v) \ : \ v \in \mathrm{AC}(0, +\infty; X), \ v(0) = u_0 \} .$$

has at least a solution u_ε.

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has at least a solution u_ε.
Indeed, let (v_n) be an infimizing sequence:

- (integral) estimate for $\varphi(v_n)$ + coercivity of $\varphi \Rightarrow$ "compactness" for (v_n)
- (integral) estimate for $|v_n'| \Rightarrow$ "equicontinuity" for (v_n)

The Euler-Lagrange equation for the WED minimization

$$u_\varepsilon \in \mathrm{Argmin}\left\{ \int_0^{+\infty} \mathrm{e}^{-t/\varepsilon} \left(\frac{1}{2}|v'|^2(t) + \frac{1}{\varepsilon}\varphi(v(t)) \right) \mathrm{d}t \ : \ v \in \mathrm{AC}(0,+\infty;X), \ v(0) = u_0 \right\}$$

- Euler-Lagrange equation in the Hilbert case:

$$-\varepsilon {u_\varepsilon}''(t) + u_\varepsilon'(t) + \partial\varphi(u_\varepsilon(t)) \ni 0 \text{ in } \mathcal{H}, \ t \in (0,+\infty) \qquad \text{(EUL)}$$

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Fact (II)

u_ε fulfils the Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\varphi(u(t))-\frac{\varepsilon}{2}|u_\varepsilon'|^2(t)\right)+|u_\varepsilon'|^2(t)=0 \quad \text{for a.a. } t\in(0,+\infty)$$

viz. the **metric version** of (EUL).

The Euler-Lagrange equation for the WED minimization

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$$\underbrace{-\varepsilon {u_\varepsilon}''(t) \times u_\varepsilon'(t)}_{\frac{\mathrm{d}}{\mathrm{d}t}(-\varepsilon\|u_\varepsilon'(t)\|^2)} + u_\varepsilon'(t) \times u_\varepsilon'(t) + \underbrace{\partial\varphi(u_\varepsilon(t)) \times u_\varepsilon'(t)}_{\frac{\mathrm{d}}{\mathrm{d}t}\varphi(u_\varepsilon(t))} = 0 \quad t \in (0,+\infty) \tag{EUL}$$

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Analytical difficulties (I)

From $\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(\varphi(u(t)) - \frac{\varepsilon}{2}|u'|^2(t)\right) + |u'|^2(t) = 0}$ we get the *energy inequality*

$$\varphi(u_\varepsilon(t)) - \frac{\varepsilon}{2}|u_\varepsilon'|^2(t) + \int_0^t |u_\varepsilon'|^2(s)\,\mathrm{d}s \leq \varphi(u_0) \quad \text{for all } t > 0 \tag{ENE}$$

which is **NOT sufficient** for taking the limit $\varepsilon \searrow 0$!!

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♠ Lack of

- **pointwise** estimates for $\varphi(u_\varepsilon)$ $\rightsquigarrow$ "compactness"
- **integral** estimates for $|u_\varepsilon'|$ $\rightsquigarrow$ "equicontinuity"

& it is not possible to reproduce in the **metric context** the estimates of **[Mielke-Stefanelli, ESAIM-COCV 2010]**!!

♢ **Problem 1:** deduce $\boxed{\text{further estimates for } u_\varepsilon}$

Analytical difficulties (II)

$$\varphi(u_\varepsilon(t)) - \frac{\varepsilon}{2}|u_\varepsilon'|^2(t) + \int_0^t |u_\varepsilon'|^2(s)\,\mathrm{d}s \leq \varphi(u_0) \quad \text{per ogni } t > 0 \qquad \text{(ENE)}$$

♠ (ENE) does not have the "right structure": to conclude that u_ε converges to a **Curve of Maximal Slope** for φ, we should pass to the limit

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BUT (ENE) does not contain information on $|\partial\varphi|(u)$!!

♢ **Problem 2:** deduce another energy inequality where we can take the limit as $\varepsilon \searrow 0$

A new viewpoint

We study the **Value Functional**

$$V_\varepsilon(\bar{u}) := \min\left\{\mathcal{J}_\varepsilon(v)\ :\ v\in \mathrm{AC}(0,+\infty;X),\ v(0)=\bar{u}\right\}$$

$$= \min\left\{\int_0^{+\infty} \mathrm{e}^{-t/\varepsilon}\left(\frac{1}{2}|v'|^2(t)+\frac{1}{\varepsilon}\varphi(v(t))\right)\mathrm{d}t\ :\ v\in \mathrm{AC}(0,+\infty;X),\ v(0)=\bar{u}\right\}$$

♣ **Remark**: $\boxed{V_\varepsilon(\bar{u}) \to \varphi(\bar{u})\ \text{ as } \varepsilon \searrow 0}$

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♣ **Remark**: $V_\varepsilon(\bar u) \to \varphi(\bar u)$ as $\varepsilon \searrow 0$

A new argument for taking the limit as $\varepsilon \searrow 0$

- Use the **Dynamic Programming Principle** for V_ε and deduce that for all $\varepsilon > 0$

  u_ε is a Curve of Maximal Slope for V_ε

- under the assumption that **φ is (geodesically)-convex** obtain further estimates for $(u_\varepsilon)_\varepsilon$
- take the limit as $\varepsilon \searrow 0$

  in the gradient flow equation of V_ε

  instead of the Euler-Lagrange equation for the WED minimization!!

The Dynamic Programming Principle in a metric context

♣ It can be shown that for all $t > 0$

$$V_\varepsilon(\bar{u}) = \min_{v \in \mathrm{AC}(0,+\infty;X),\ v(0)=\bar{u}} \left[\int_0^t \mathrm{e}^{-s/\varepsilon} \left(\frac{1}{2}|v'|^2(s) + \frac{1}{\varepsilon}\varphi(v(s)) \right) \mathrm{d}s + V_\varepsilon(v(t))\mathrm{e}^{-t/\varepsilon} \right]$$

viz., to achieve the minimum cost $V_\varepsilon(\bar{u})$ it is necessary & sufficient to:

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4. **minimize** over all possible trajectories

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whence, for all $t > 0$,

$$-\int_0^t \frac{\mathrm{d}}{\mathrm{d}t}(V_\varepsilon(u_\varepsilon(s)))\mathrm{e}^{-s/\varepsilon}\,\mathrm{d}s = \int_0^t \mathrm{e}^{-s/\varepsilon} \left(\frac{1}{2}|u_\varepsilon'|^2(s) + \frac{1}{\varepsilon}\varphi(u_\varepsilon(s)) - \frac{1}{\varepsilon}V_\varepsilon(u_\varepsilon(s)) \right) \mathrm{d}s$$

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$$\boxed{V_\varepsilon(\bar{u}) = \min_{v \in \mathrm{AC}(0,+\infty;X),\ v(0)=\bar{u}} \left[\int_0^t \mathrm{e}^{-s/\varepsilon} \left(\frac{1}{2}|v'|^2(s) + \frac{1}{\varepsilon}\varphi(v(s)) \right) \mathrm{d}s + V_\varepsilon(v(t))\mathrm{e}^{-t/\varepsilon} \right]}$$

hence $u_\varepsilon \in \mathrm{Argmin}_{v(0)=u_0} \mathcal{J}_\varepsilon(v)$ fulfils

$$\underbrace{V_\varepsilon(u_\varepsilon(0)) - V_\varepsilon(u_\varepsilon(t))\mathrm{e}^{-t/\varepsilon}}_{= -\int_0^t \frac{\mathrm{d}}{\mathrm{d}t}\left(V_\varepsilon(u_\varepsilon(s))\mathrm{e}^{-s/\varepsilon}\right)\mathrm{d}s} = \int_0^t \mathrm{e}^{-s/\varepsilon} \left(\frac{1}{2}|u_\varepsilon'|^2(s) + \frac{1}{\varepsilon}\varphi(u_\varepsilon(s)) \right) \mathrm{d}s \quad \text{for all } t > 0$$

whence, for all $t > 0$,

$$-\int_0^t \frac{\mathrm{d}}{\mathrm{d}t}(V_\varepsilon(u_\varepsilon(s)))\mathrm{e}^{-s/\varepsilon}\,\mathrm{d}s = \int_0^t \mathrm{e}^{-s/\varepsilon} \left(\frac{1}{2}|u_\varepsilon'|^2(s) + \frac{1}{\varepsilon}\varphi(u_\varepsilon(s)) - \frac{1}{\varepsilon}V_\varepsilon(u_\varepsilon(s)) \right) \mathrm{d}s$$

hence the identity

$$\boxed{-\frac{\mathrm{d}}{\mathrm{d}t}V_\varepsilon(u_\varepsilon(t)) = \frac{1}{2}|u_\varepsilon'|^2(s) + \frac{1}{\varepsilon}\varphi(u_\varepsilon(t)) - \frac{1}{\varepsilon}V_\varepsilon(u_\varepsilon(t)) \quad \text{for a.a. } t \in (0,T).}$$

The Hamilton-Jacobi-Bellman equation in a metric context

For all $\varepsilon > 0$ there holds

$$\boxed{\frac{1}{\varepsilon}\varphi(\bar{u}) - \frac{1}{\varepsilon}V_\varepsilon(\bar{u}) = \frac{1}{2}|\partial V_\varepsilon|^2(\bar{u}) \quad \text{for all } \bar{u} \in X} \tag{H-J}$$

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♣ Let's prove $\boxed{\leq}$ with the **Dynamic Programming Principle**: for any curve $u_{\min} \in \mathrm{Argmin}_{v(0)=\bar{u}} \mathcal{J}_\varepsilon(v)$ there holds

$$V_\varepsilon(\bar{u}) - V_\varepsilon(u_{\min}(\delta))\mathrm{e}^{-\delta/\varepsilon} = \int_0^\delta \mathrm{e}^{-t/\varepsilon}\left(\frac{1}{2}|u'_{\min}|^2(t) + \frac{1}{\varepsilon}\varphi(u_{\min}(t))\right)\mathrm{d}t$$

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hence

$$\frac{1}{\delta}\int_0^\delta \mathrm{e}^{-t/\varepsilon}\left(\frac{1}{2}|u'_{\min}|^2(t) + \frac{1}{\varepsilon}\varphi(u_{\min}(t))\right)\,\mathrm{d}t - \frac{1-e^{-\delta/\varepsilon}}{\delta}V_\varepsilon(u_{\min}(\delta)) = \frac{V_\varepsilon(\bar{u}) - V_\varepsilon(u_{\min}(\delta))}{\delta}.$$

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and then we take the $\boxed{\lim_{\delta \searrow 0}}$ for $\boxed{\text{fixed } \varepsilon > 0}$.

The Hamilton-Jacobi-Bellman equation in a metric context

$$\boxed{\frac{1}{\varepsilon}\varphi(\bar{u}) - \frac{1}{\varepsilon}V_\varepsilon(\bar{u}) = \frac{1}{2}|\partial V_\varepsilon|^2(\bar{u}) \quad \text{for all } \bar{u} \in X} \tag{H-J}$$

There holds

$$\underbrace{\liminf_{\delta \searrow 0} \frac{1}{\delta}\int_0^\delta e^{-t/\varepsilon}\left(\frac{1}{2}|u'_{\min}|^2(t) + \frac{1}{\varepsilon}\varphi(u_{\min}(t))\right)\mathrm{d}t}_{\geq \frac{1}{2}|u'_{\min}|^2(0) + \frac{1}{\varepsilon}\varphi(\bar{u})} \quad \underbrace{-\lim_{\delta \searrow 0}\frac{1-e^{-\delta/\varepsilon}}{\delta}V_\varepsilon(u_{\min}(\delta))}_{= -\frac{1}{\varepsilon}V_\varepsilon(\bar{u})}$$

$$\leq \limsup_{\delta \searrow 0} \frac{V_\varepsilon(\bar{u}) - V_\varepsilon(u_{\min}(\delta))}{\delta}$$

$$\leq \limsup_{\delta \searrow 0} \frac{(V_\varepsilon(\bar{u}) - V_\varepsilon(u_{\min}(\delta)))^+}{d(u_{\min}(\delta), \bar{u})}\,\frac{d(u_{\min}(\delta), \bar{u})}{\delta} \leq |\partial V_\varepsilon|(\bar{u})|u'_{\min}|(0)$$

Hence

$$\frac{1}{2}|u'_{\min}|^2(0) + \frac{1}{\varepsilon}\varphi(\bar{u}) - \frac{1}{\varepsilon}V_\varepsilon(\bar{u}) \leq |\partial V_\varepsilon|(\bar{u})|u'_{\min}|(0)$$

whence

$$\boxed{\frac{1}{\varepsilon}\varphi(\bar{u}) - \frac{1}{\varepsilon}V_\varepsilon(\bar{u}) \leq \frac{1}{2}|\partial V_\varepsilon|^2(\bar{u})}$$

The Hamilton-Jacobi-Bellman equation in a metric context

For all $\varepsilon > 0$ there holds

$$\boxed{\frac{1}{\varepsilon}\varphi(\bar{u}) - \frac{1}{\varepsilon}V_\varepsilon(\bar{u}) = \frac{1}{2}|\partial V_\varepsilon|^2(\bar{u}) \quad \text{for all } \bar{u} \in X} \tag{H-J}$$

- ♣ The proof of $\boxed{\leq}$ only uses lower semicontinuity and coercivity of φ & the Dynamic Programming Principle
- ♠ The proof of $\boxed{\geq}$ instead relies on

$\boxed{\varphi \text{ **(geodesically-)convex** on } X}$

The gradient flow of the WED minimizers

♣ Let $u_\varepsilon \in \mathrm{Argmin}_{v(0)=u_0} \mathcal{J}_\varepsilon(v)$: Combining

$$-\frac{\mathrm{d}}{\mathrm{d}t} V_\varepsilon(u_\varepsilon(t)) = \frac{1}{2}|u_\varepsilon'|^2(s) + \frac{1}{\varepsilon}\varphi(u_\varepsilon(t)) - \frac{1}{\varepsilon}V_\varepsilon(u_\varepsilon(t)) \quad \text{for a.a. } t \in (0,T).$$

with the Hamilton-Jacobi-Bellman equation

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we conclude

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viz. for all $\varepsilon > 0$, u_ε **is a Curve of Maximal Slope for the Value functional** V_ε.

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✓ It's in (EQV) that one has to take the limit as $\varepsilon \searrow 0$!!!

Energy estimates in the convex case

Assumptions on φ

- $\varphi : X \to (-\infty, +\infty]$ positive, l.s.c., **coercive**,
- $\varphi : X \to (-\infty, +\infty]$ (geodesically-)**convex**

Then $u_\varepsilon \in \mathrm{Argmin}_{v(0)=u_0} \mathcal{J}_\varepsilon(v)$ has the further properties:

- $t \mapsto |u'_\varepsilon|(t)$ is decreasing;
- $t \mapsto \varphi(u_\varepsilon(t))$ is convex;

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- $t \mapsto |u'_\varepsilon|(t)$ is decreasing;
- $t \mapsto \varphi(u_\varepsilon(t))$ is convex;
- $t \mapsto \varphi(u_\varepsilon(t))$ is decreasing: hence we get an **energy estimate**

$$\sup_{\varepsilon>0,\, t>0} \varphi(u_\varepsilon(t)) \leq \varphi(u_0)$$

From the gradient flow of V_ε to the gradient flow of φ

- **Passage to the limit as $\varepsilon \searrow 0$** in

$$\boxed{-\frac{\mathrm{d}}{\mathrm{d}t} V_\varepsilon(u_\varepsilon(t)) = \frac{1}{2}|u_\varepsilon'|^2(s) + \frac{1}{2}|\partial V_\varepsilon|^2(u_\varepsilon(t))}$$

i.e. in

$$\frac{1}{2}\int_0^t |u_\varepsilon'|^2(s)\,\mathrm{d}s + \frac{1}{2}\int_0^t |\partial V_\varepsilon|^2(u_\varepsilon(s))\,\mathrm{d}s + V_\varepsilon(u_\varepsilon(t)) = V_\varepsilon(u_0)$$

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  - **A priori estimates:**

$$\sup_{\varepsilon>0,\, t>0} \varphi(u_\varepsilon(t)) \leq \varphi(u_0) \qquad \rightsquigarrow \qquad \text{compactness}$$

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  - **Ascoli-Arzelà Theorem** (in metric spaces):

$$\exists\,(u_{\varepsilon_k})_k,\ \exists\, u \in \mathrm{AC}(0,+\infty;X)\ :\quad u_{\varepsilon_k}(t) \to u(t) \quad \forall\, t \in (0,+\infty).$$

From the gradient flow of V_ε to the gradient flow of φ

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♣ It can be checked that

$$\begin{aligned} \frac{1}{2}\int_0^t |u'|^2(s)\,\mathrm{d}s &\leq \liminf_{k\to\infty} \frac{1}{2}\int_0^t |u_{\varepsilon_k}'|^2(s)\,\mathrm{d}s \\ \varphi(u(t)) &\leq \liminf_{k\to\infty} V_{\varepsilon_k}(u_{\varepsilon_k}(t)), \quad V_{\varepsilon_k}(u_0) \to \varphi(u_0) \end{aligned}$$

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♣ It can also be proved that

$$\frac{1}{2}\int_0^t |\partial\varphi|^2(u(s))\,\mathrm{d}s \leq \liminf_{k\to\infty} \frac{1}{2}\int_0^t |\partial V_\varepsilon|^2(u_{\varepsilon_k}(s))\,\mathrm{d}s$$

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- Hence $u \in \mathrm{AC}(0,+\infty;X)$ fulfils for all $t \in (0,+\infty)$

$$\frac{1}{2}\int_0^t |u'|^2(s)\,\mathrm{d}s + \frac{1}{2}\int_0^t |\partial\varphi|^2(u(s))\,\mathrm{d}s + \varphi(u(t)) \leq \varphi(u_0)$$

equivalent (via chain rule) to $\boxed{\frac{1}{2}|u'|^2(t) + \frac{1}{2}|\partial\varphi|^2(u(t)) = -\frac{\mathrm{d}}{\mathrm{d}t}\varphi(u(t)) \quad \text{a.e. in } (0,+\infty)}$

The main result

Theorem (R., Savaré, Segatti, Stefanelli 2011)

Suppose that $\varphi : X \to (-\infty, +\infty]$ is

- positive, l.s.c., **coercive**,
- (geodesically-)**convex**;

let $u_0 \in D(\varphi)$.
Then,

any sequence of WED minimizers (u_ε) with $u_\varepsilon(0) = u_0$
has a subsequence (u_{ε_k}) which converges as $\varepsilon_k \searrow 0$ to
$u \in \mathrm{AC}(0, +\infty; X)$, Curve of Maximal Slope for φ such that $u(0) = u_0$.

Conclusions & future developments

- φ (geodesically-)convex $\rightsquigarrow$ **φ (geodesically-) λ-convex, with $\lambda < 0$**
- u_ε WED minimizer $\rightsquigarrow$ **u_ε approximate minimizer for the WED functional**
- combine **WED approximation with time-discretization** (Minimizing Movements)
- ....

Future development: φ **NOT (geodesically-) λ-convex**..... it's an open problem in the Hilbert case as well!