Analysis of a model for adhesive contact with thermal effects

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joint work with
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Frémond's modeling of adhesive contact

Setting

A viscoelastic body $\Omega \subset \mathbb{R}^3$ in **adhesive contact** with a rigid support on a **(flat)** part Γ_{Cont} of its boundary $\partial\Omega = \Gamma_{\mathrm{Dir}} \cup \Gamma_{\mathrm{Neu}} \cup \Gamma_{\mathrm{Cont}}$.

Related contributions

.... Andrews, Cangémi, Chau, Cocou, Eck, Fernández, Figuereido, Han, Jarušek, Klarbring, Krbec, Kuttler, Martins, Muñoz-Rivera, Point, Racke, Raous, Shi, Shillor, Sofonea, Telega, Trabucho, Wright.....

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Based on **[M. Frémond, *Non-smooth Thermomechanics*, 2002]**

Account for **microscopic motions** in the **macroscopic predictive theory**

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Account for **microscopic motions** in the **macroscopic predictive theory**

- microscopic bonds are responsible for the adhesion, microscopic motions lead to rupture
- account for the **power of the microscopic motions** in the power of the interior forces

State variables

In the **isothermal case** **[Bonetti-Bonfanti-R. '07,'08]**

- in the **volume domain** Ω:
  - **small deformation** ($\varepsilon(\mathbf{u})$ symm. linear. strain tensor) **(small perturbation assumption)**
  -
- on the **contact surface** Γ_{Cont}:
  - **adhesion** (χ "phase parameter" related to the active bonds of the adhesion $\rightsquigarrow$ "damage parameter")
  - **effects of displacement** ($\mathbf{u}_{|\Gamma_{\mathrm{Cont}}}$ trace of the displacement)
  -

State variables

To account for **thermal effects:** **[Bonetti-Bonfanti-R. preprint'08]**

- in the **volume domain** Ω:
  - **small deformation** ($\varepsilon(\mathbf{u})$ symm. linear. strain tensor) **(small perturbation assumption)**
  - **thermal effects (θ absolute temperature)**
- on the **contact surface** Γ_{Cont}:
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The equations for u and χ

From the **principle of virtual power** (interior & exterior forces, no acceleration forces)

- **momentum balance:**

$$\begin{cases} -\operatorname{div}\Sigma = \mathbf{f} & \text{in } \Omega \times (0,T), \\ \begin{cases} \Sigma\mathbf{n} = \mathbf{R} & \text{in } \Gamma_{\mathrm{Cont}} \times (0,T), \\ \mathbf{u} = \mathbf{0} & \text{in } \Gamma_{\mathrm{Dir}} \times (0,T), \\ \Sigma\mathbf{n} = \mathbf{g} & \text{in } \Gamma_{\mathrm{Neu}} \times (0,T), \end{cases} \end{cases}$$

$$\begin{cases} \Sigma \text{ stress tensor} \\ \mathbf{R} \text{ reaction on the contact surface} \\ \mathbf{f} \text{ volume force, } \mathbf{g} \text{ traction} \end{cases}$$

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- **equation for the microscopic motions:**

$$\begin{cases} B - \operatorname{div}\mathbf{H} = 0 & \text{in } \Gamma_{\mathrm{Cont}}\times(0,T), \\ \mathbf{H}\cdot\mathbf{n}_{\mathrm{s}} = 0 & \text{on } \partial\Gamma_{\mathrm{Cont}}\times(0,T), \end{cases}$$

$$\begin{cases} B & \text{interior microscopic work} \\ \mathbf{H} & \text{microscopic work flux vector} \end{cases}$$

The equations for θ and θ_{s}

Entropy balance for θ and θ_{s}:

- for θ:

$$\begin{cases} s_t + \operatorname{div} \mathbf{Q} = h \quad \text{in } \Omega \times (0,T), \\ \begin{cases} \mathbf{Q} \cdot \mathbf{n} = F \ \text{ on } \Gamma_{\mathrm{Cont}} \times (0,T), \\ \quad \mathbf{Q} \cdot \mathbf{n} = 0 \ \text{ on } \partial\Omega \setminus \Gamma_{\mathrm{Cont}} \times (0,T), \end{cases} \end{cases}$$

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- for θ_{s} :

$$\begin{cases} \partial_t s_{\mathrm{s}} + \operatorname{div} \mathbf{Q}_{\mathrm{s}} = F & \text{in } \Omega \times (0,T), \\ \mathbf{Q}_{\mathrm{s}} \cdot \mathbf{n}_{\mathrm{s}} = 0 & \text{on } \partial\Gamma_{\mathrm{Cont}} \times (0,T), \end{cases}$$

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Entropy balance: obtained by rescaling the internal energy balance (under small perturbation assumpt.): see **[Bonetti-Frémond'03, Bonetti-Colli-Frémond'03, Bonetti'06, Bonetti-Colli-Fabrizio-Gilardi'06,'07,08, Bonetti-Rocca-Frémond'07]**

Constitutive laws

Constitutive relations for

$$\Sigma, \mathbf{R}, s, \mathbf{Q}, F, B, \mathbf{H}, s_{\mathrm{s}}, \mathbf{Q}_{\mathrm{s}}$$

derive from the **volume & surface free energies**

$$\Psi_{\Omega} = \Psi_{\Omega}(\mathbf{u}, \theta), \qquad \Psi_{\Gamma_{\mathrm{Cont}}} = \Psi_{\Gamma_{\mathrm{Cont}}}(\mathbf{u}_{|\Gamma_{\mathrm{Cont}}}, \chi, \theta_{\mathrm{s}})$$

and the **pseudo-potentials of dissipation**

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The energy potentials include **constraints on the variables** for physical consistency: $\rightsquigarrow$ **nonsmooth (multivalued) operators** in the constitutive eqns

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Constraints

- admissible values for χ and (possibly) irreversibility
- impenetrability condition between the body and the support
- positivity of the absolute temperatures θ and θ_{s}

The adhesion phenomenon

The "damage parameter" χ denotes the **fraction of active glue fibers** at each point of the contact surface

- $\chi = 0$ no adhesion (completely broken bonds)
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Physical constraints

- $\chi \in [0, 1]$
- $\chi_t \leq 0$ (irreversible phenomenon)

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Physical constraints

- $\chi \in [0, 1]$

As a first step, neglect irreversibility ($\sim$ fresh, liquid glue..)

Unilateral conditions

The **impenetrability condition**

$$\mathbf{u}_{|\Gamma_{\mathrm{Cont}}} \cdot \mathbf{n} \leq \mathbf{0} \quad \text{on } \Gamma_{\mathrm{Cont}}.$$

is ensured by the reaction on the contact surface

Unilateral conditions

The **impenetrability condition**

$$\mathbf{u}_{|\Gamma_{\mathrm{Cont}}} \cdot \mathbf{n} \leq \mathbf{0} \quad \text{on } \Gamma_{\mathrm{Cont}}.$$

is ensured by the reaction on the contact surface
If $\chi = 0$ (no adhesion) the reaction on Γ_{Cont} is

$$\mathbf{R} = \qquad -\partial I_{]-\infty,0]}(\mathbf{u} \cdot \mathbf{n})\mathbf{n}$$

(omitting traces) where

$$\partial I_{(-\infty,0]}(\mathbf{u} \cdot \mathbf{n}) = \begin{cases} 0 & \text{if } \mathbf{u} \cdot \mathbf{n} < \mathbf{0} \\ [0, +\infty[& \text{if } \mathbf{u} \cdot \mathbf{n} = \mathbf{0} \end{cases}$$

- $\mathbf{R}$ is normal to Γ_{Cont}
- $\mathbf{R} = \mathbf{0}$ if $\mathbf{u} \cdot \mathbf{n} < 0$
- $\mathbf{R} = \gamma \mathbf{n}$, $\gamma \leq 0$ if $\mathbf{u} \cdot \mathbf{n} = 0$

This is in agreement with the **Signorini conditions**

Unilateral conditions

The **impenetrability condition**

$$\mathbf{u}_{|\Gamma_{\mathrm{Cont}}} \cdot \mathbf{n} \leq \mathbf{0} \quad \text{on } \Gamma_{\mathrm{Cont}}.$$

is ensured by the reaction on the contact surface
In the case the adhesion is active $\chi > 0$

$$\mathbf{R} = -\chi \mathbf{u} - \partial \mathbf{I}_{]-\infty,\mathbf{0}]}(\mathbf{u} \cdot \mathbf{n})\mathbf{n}$$

i.e., there is a reaction (with rigidity $\sim \chi$) **counteracting separation**:

$$\mathbf{R} \cdot \mathbf{n} = -\chi\, \mathbf{u} \cdot \mathbf{n} > 0 \text{ if } \mathbf{u} \cdot \mathbf{n} < 0$$

The PDE system: the momentum balance

Recall $\Omega \subset \mathbb{R}^3$ smooth, bounded and $\partial\Omega = \Gamma_{\mathrm{Dir}} \cup \Gamma_{\mathrm{Neu}} \cup \Gamma_{\mathrm{Cont}}$

- The momentum balance

$$-\mathrm{div}\,(K\varepsilon(\mathbf{u}) + K_v\varepsilon(\mathbf{u}_t) + \theta\mathbf{1}) = \mathbf{f} \quad \text{in } \Omega \times (0, T)$$

where: K elasticity tensor, K_v viscosity tensor, $\theta\mathbf{1} \leftrightarrow$ **thermal deformation**

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- the **boundary conditions**

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_{\mathrm{Dir}}, \quad (K\varepsilon(\mathbf{u}) + K_v\varepsilon(\mathbf{u}_t) + \theta\mathbf{1})\mathbf{n} = \mathbf{g} \quad \text{on } \Gamma_{\mathrm{Neu}} \times (0,T),$$

$$(K\varepsilon(\mathbf{u}) + K_v\varepsilon(\mathbf{u}_t) + \theta\mathbf{1})\mathbf{n} + \chi\mathbf{u} + \partial I_{]-\infty,0]}(\mathbf{u}\cdot\mathbf{n})\mathbf{n} \ni \mathbf{0} \quad \text{on } \Gamma_{\mathrm{Cont}} \times (0,T),$$

where $-\chi\mathbf{u} - \partial I_{]-\infty,0]}(\mathbf{u}\cdot\mathbf{n})\mathbf{n}$ on Γ_{Cont} is the reaction

The PDE system: the evolution of the adhesion

We consider on Γ_{Cont}

$$\chi_t - \Delta\chi + \partial \mathbf{I}_{[0,1]}(\chi) \ni -\lambda'(\chi)(\theta_{\mathrm{s}} - \theta_{\mathbf{eq}}) - \frac{1}{2}|\mathbf{u}|^2 \qquad \text{in } \Gamma_{\mathrm{Cont}} \times (0,T)$$

$$\partial_n \chi = 0, \quad \text{on } \partial\Gamma_{\mathrm{Cont}} \times (0,T)$$

- $\partial \mathbf{I}_{[0,1]}(\chi) \Rightarrow \chi \in [0,1]$ **(physical consistency)**

$$\partial I_{[0,1]}(\chi) = \begin{cases}]-\infty, 0] & \text{if } \chi = 0 \\ 0 & \text{if } 0 < \chi < 1 \\ [0, +\infty[& \text{if } \chi = 1 \end{cases}$$

- λ (quadratic) function, related to the latent heat
- θ_{s} temperature of the glue, θ_{eq} constant
- $-\frac{1}{2}|\mathbf{u}|^2$ source of damage due to macroscopic movements

The PDE system: the temperature equations

- The **entropy equation** (rescaled energy balance) for θ in Ω

$$\partial_t(\log\theta) - \mathrm{div}\,\mathbf{u}_t - \Delta\theta = h \quad \text{in } \Omega \times (0,T),$$

$$\partial_n\theta = \begin{cases} 0 & \text{on } \partial\Omega \setminus \Gamma_{\mathrm{Cont}} \times (0,T) \\ -\chi(\theta_{|\Gamma_{\mathrm{Cont}}} - \theta_{\mathrm{s}}) & \text{on } \Gamma_{\mathrm{Cont}} \times (0,T) \end{cases}$$

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- The **entropy equation** for θ_{s} on Γ_{Cont} is

$$\partial_t(\log\theta_{\mathrm{s}}) - \lambda'(\chi)\chi_t - \Delta\theta_{\mathrm{s}} = \chi(\theta_{|\Gamma_{\mathrm{Cont}}} - \theta_{\mathrm{s}}) \quad \text{in } \Gamma_{\mathrm{Cont}} \times (0,T)$$

$$\partial_n\theta_{\mathrm{s}} = 0 \quad \text{on } \partial\Gamma_{\mathrm{Cont}} \times (0,T)$$

The complete PDE system: difficulties

$$
\begin{aligned}
& -\operatorname{div}\left(K\varepsilon(\mathbf{u})+K_v\varepsilon(\mathbf{u}_t)+\theta\mathbf{1}\right)=\mathbf{f} \quad \text{in } \Omega\times(0,T),\\
& \mathbf{u}=\mathbf{0} \quad \text{on } \Gamma_{\mathrm{Dir}}\times(0,T), \quad \left(K\varepsilon(\mathbf{u})+K_v\varepsilon(\mathbf{u}_t)+\theta\mathbf{1}\right)\mathbf{n}=\mathbf{g} \quad \text{on } \Gamma_{\mathrm{Neu}}\times(0,T),\\
& \left(K\varepsilon(\mathbf{u})+K_v\varepsilon(\mathbf{u}_t)+\theta\mathbf{1}\right)\mathbf{n}+\chi\mathbf{u}+\partial I_{]-\infty,0]}(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\ni\mathbf{0} \quad \text{on } \Gamma_{\mathrm{Cont}}\times(0,T),\\
& \chi_t-\Delta\chi+\partial I_{[0,1]}(\chi)\ni-\lambda'(\chi)(\theta_s-\theta_{eq})-\frac{1}{2}|\mathbf{u}|^2 \quad \text{in } \Gamma_{\mathrm{Cont}}\times(0,T),\\
& \partial_n\chi=0 \quad \text{on } \partial\Gamma_{\mathrm{Cont}}\times(0,T)\\
& \partial_t(\log\theta)-\operatorname{div}\mathbf{u}_t-\Delta\theta=h \quad \text{in } \Omega\times(0,T),\\
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& \partial_t(\log\theta_s)-\lambda'(\chi)\chi_t-\Delta\theta_s=\chi(\theta-\theta_s) \quad \text{in } \Gamma_{\mathrm{Cont}}\times(0,T),\\
& \partial_n\theta_s=0 \quad \text{on } \partial\Gamma_{\mathrm{Cont}}\times(0,T), \qquad +\text{Cauchy conditions}
\end{aligned}
$$

→ **singular character of the θ, θ_s-equations** (θ-**equation is coupled with a third type boundary condition**)
→ we deduce directly $\theta, \theta_s > 0$, crucial for **thermodynamical consistency**!

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& \mathbf{u} = \mathbf{0} \quad \text{in } \Gamma_{\mathrm{Dir}} \times (0,T), \quad (K\varepsilon(\mathbf{u}) + K_v\varepsilon(\mathbf{u}_t) + \theta\mathbf{1})\mathbf{n} = \mathbf{g} \quad \text{in } \Gamma_{\mathrm{Neu}} \times (0,T),\\
& (K\varepsilon(\mathbf{u}) + K_v\varepsilon(\mathbf{u}_t) + \theta\mathbf{1})\mathbf{n} + \chi\mathbf{u} + \partial I_{]-\infty,0]}(\mathbf{u}\cdot\mathbf{n})\mathbf{n} \ni \mathbf{0} \quad \text{in } \Gamma_{\mathrm{Cont}} \times (0,T),\\
& \chi_t - \Delta\chi + \partial I_{[0,1]}(\chi) \ni -\lambda'(\chi)(\theta_s - \theta_{eq}) - \frac{1}{2}|\mathbf{u}|^2 \quad \text{in } \Gamma_{\mathrm{Cont}} \times (0,T),\\
& \partial_n\chi = 0 \quad \text{on } \partial\Gamma_{\mathrm{Cont}} \times (0,T)\\
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→ **(quadratic) coupling** terms on the boundary
→ we need **sufficient regularity** on θ and $\mathbf{u}$ to control their traces.

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& \left(K\varepsilon(\mathbf{u}) + K_v\varepsilon(\mathbf{u}_t) + \theta\mathbf{1}\right)\mathbf{n} + \chi\mathbf{u} + \partial I_{]-\infty,0]}(\mathbf{u}\cdot\mathbf{n})\mathbf{n} \ni \mathbf{0} \quad \text{on } \Gamma_{\mathrm{Cont}} \times (0,T), \\
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& \partial_n\theta = \begin{cases} 0 & \text{on } \partial\Omega \setminus \Gamma_{\mathrm{Cont}} \times (0,T), \\ -\chi(\theta - \theta_s) & \text{on } \Gamma_{\mathrm{Cont}} \times (0,T), \end{cases} \\
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& \partial_n\theta_s = 0 \quad \text{on } \partial\Gamma_{\mathrm{Cont}} \times (0,T), \qquad +\text{Cauchy conditions}
\end{aligned}
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$\rightarrow$ **non-smooth (multivalued) constraints** on χ and $\mathbf{u}\cdot\mathbf{n}$

The existence theorem

Theorem. Given initial data

$$\begin{cases} \theta_0 \in L^1(\Omega), \quad \log(\theta_0) \in L^2(\Omega), \quad \theta_{s,0} \in L^1(\Gamma_{\mathrm{Cont}}), \quad \log(\theta_{s,0}) \in L^2(\Gamma_{\mathrm{Cont}}), \\ \mathbf{u}_0 \in H^1(\Omega)^3, \quad \chi_0 \in H^1(\Gamma_{\mathrm{Cont}}) \end{cases}$$

there exist $(\mathbf{u}, \chi, \theta, \theta_s)$ solving the **weak, variational** formulation of the **IBVP**

$$-\operatorname{div}\left(K\varepsilon(\mathbf{u}) + K_v\varepsilon(\mathbf{u}_t) + \theta\mathbf{1}\right) = \mathbf{f} \quad \text{in } \Omega \times (0,T),$$

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- we do not approximate $\partial I_{[0,1]}(\chi)$, $\partial I_{]-\infty,0]}(\mathbf{u}\cdot\mathbf{n})$ in the eq.'s for $\mathbf{u}$ and χ

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Uniqueness?

- **NOT expected** due to the nonlinear structure of the equations, the lack of regularity of θ and the boundary conditions
- holds for the **approximating problem**

Trajectories on $(0, +\infty)$

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How the trajectories $(\mathbf{u}(t), \chi(t), \theta(t), \theta_s(t))$ behave as $t \to +\infty$?

- need **uniform estimates** on the solutions independent of the final time T
- **further requirement** on the entropy flux through Γ_c

$$\partial_n \theta = -(\chi + c)(\theta_{|\Gamma_c} - \theta_s)\,, \quad c > 0 \text{ on } \Gamma_c$$

$\rightsquigarrow$ residual flux even if $\chi = 0$
(crucial to obtain a L^2-estimate for $\theta_{|\Gamma_c} - \theta_s$)

Long-time a priori estimates

Due to the **dissipative character of the system**

$$
\begin{aligned}
&|\theta|_{L^\infty(1,+\infty;H^1(\Omega))} + |\nabla\theta|_{L^2(0,+\infty;L^2(\Omega))} \leq C \\
&|\theta_t|_{L^2(1,+\infty;L^p(\Omega))} \leq C \quad \text{for } p \leq 12/7 \\
&|\theta_s|_{L^\infty(1,+\infty;H^1(\Gamma_{\mathrm{Cont}}))} + |\nabla\theta_s|_{L^2(0,+\infty;L^2(\Gamma_{\mathrm{Cont}}))} \leq C \\
&|\partial_t\theta_s|_{L^2(1,+\infty;L^q(\Gamma_{\mathrm{Cont}}))} \leq C, \quad \text{for } q < 2 \\
&|\theta_{|\Gamma_{\mathrm{Cont}}} - \theta_s|_{L^2(0,+\infty;L^2(\Gamma_{\mathrm{Cont}}))} \leq C \\
&|\chi|_{L^\infty(1,+\infty;H^2(\Gamma_{\mathrm{Cont}}))} + |\chi_t|_{L^2(1,+\infty;H^1(\Gamma_{\mathrm{Cont}}))} \leq C \\
&|\mathbf{u}|_{L^\infty(0,+\infty;H^1(\Omega)^3)} + |\mathbf{u}_t|_{L^2(0,+\infty;H^1(\Omega)^3)} \leq C \\
&|\partial_t(\log\theta)|_{L^2(0,+\infty;H^1(\Omega)')} + |\partial_t(\log\theta_s)|_{L^2(0,+\infty;H^1(\Gamma_{\mathrm{Cont}})')} \leq C
\end{aligned}
$$

⇝ the **"energy"** and the **"dissipation"** are **uniformly bounded**

⇝ the solutions trajectories **converge** in a suitable sense to some cluster points as $t\infty$

The ω-limit set

The set of the **possible cluster points** $(\mathbf{u}_\infty, \chi_\infty, \theta_\infty, \theta_{s\infty})$ of the solutions trajectories

- is non-empty, connected and compact (w.r.t. to a suitable topology)

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- thermomechanical equilibrium (**no dissipation**) in the limit $t \to \infty$.