EVOLUTION PROBLEMS
IN MEMORY OF BRUNELLO TERRENI
Rapallo, 26-27 March 2004

COMPACTNESS RESULTS FOR EVOLUTION EQUATIONS

RICCARDA ROSSI
in collaboration with GIUSEPPE SAVARÉ
Dipartimento di Matematica "F.Casorati",
Università di Pavia.

COMPACTNESS IN L^p SPACES

$$\mathcal{U} \subset L^p(0,T;B),$$
B separable Banach space,
$$1 \le p < \infty.$$

Problem: Find **necessary** and **sufficient** conditions for

$\mathcal{U} \subset L^p(0,T;B)$ relatively compact.

Crucial issue for evolutionary PDEs: compactness method for nonlinear evolution equations [Lions'69, Lions-Magenes'72].

THE RIESZ-FRÉCHET-KOLMOGOROV CRITERION

B **finite** dimensional, $\{u_n\}$ bounded in $L^p(0,T;B)$.

$\{u_n\}$ is relatively compact in $L^p(0,T;B)$ **iff**

$$\lim_{h\downarrow 0} \sup_{n\in\mathbb{N}} \int_0^{T-h} \|u_n(t+h)-u_n(t)\|_B^p \, dt = 0. \tag{IEC}$$

(INTEGRAL EQUICONTINUITY CONDITION)

(IEC) holds if e.g. $\{u_n\}$ is bounded in $W^{1,p}(0,T;B)$.

When B is **infinite** dimensional,

(IEC) is **not sufficient** for L^p compactness!

$\Rightarrow$ Need for a **compactness** condition **on the values** of $\{u_n\}$.

THE AUBIN-LIONS CONDITION

[Lions'61], [Aubin'63]: Let $\{u_n\} \subset L^p(0,T;B)$ fulfil (IEC) and $\exists$ a Banach space $X \subset B$ with *compact* injection s.t.

$$\sup_{n\in\mathbb{N}} \int_0^T \|u_n(t)\|_X^p \, dt < +\infty. \tag{SC1}$$

Then, $\{u_n\}$ is relatively compact in $L^p(0,T;B)$.

- We also have

$X \subset B$ with compact injection,

$B \subset Y$ with continuous injection.

$\Rightarrow L^p(0,T;X) \cap W^{1,p}(0,T;Y) \subset L^p(0,T;B)$ with compact injection.

SIMON'S INTEGRAL CHARACTERIZATION

[Simon'87]: $\{u_n\}$ is relatively compact in $L^p(0,T;B)$ **iff** $\{u_n\}$ fulfils (IEC) and

$$\left\{ \int_0^t u_n(s)ds \ : \ n \in \mathbb{N} \right\}$$

is relatively compact in $B \quad \forall\, t \in (0,T)$. (SC2)

Proof: approximation by convolution + Ascoli-Arzela's compactness theorem.

COMPACTNESS IN TIME AND COMPACTNESS IN SPACE

Two main **ingredients** in these **compactness** criteria:

- **Compactness in time:** via the INTEGRAL EQUICONTINUITY CONDITION;
- **Compactness in space:** (when B is not finite dimensional) a compactness condition

(SC1): on the values of $\{u_n\}$, ([A., L.], only **sufficient**)

(SC2): on the time integrals of $\{u_n\}$, ([S.], **necessary & sufficient**).

(SC2) is **weaker** than (SC1); (SC1) is **easier** to handle in the **applications.**

MAIN ISSUES

- **Necessity:** Is the Aubin-Lions condition necessary?

- **Generalizations:**

  - of the **Aubin-Lions** condition:

$$\sup_{n\in\mathbb{N}} \int_0^T \|u_n(t)\|_X \, dt < +\infty \quad \rightsquigarrow \quad \sup_{n\in\mathbb{N}} \int_0^T \mathscr{F}(u_n(t)) \, dt < +\infty ?$$

for a suitable functional $\mathscr{F}$

  - of the **integral equicontinuity** condition: e.g., replace

$$\| \cdot \|_B \quad \rightsquigarrow \quad d_B(\cdot, \cdot)?$$

(when (B, d_B) is a complete, separable, **metric** space only).

FROM L^p COMPACTNESS
TO COMPACTNESS IN MEASURE

Definitions:

- $\{u_n\}$, $u \in \boldsymbol{\mathcal{M}(0,T;B)}$ $(\mathcal{M}(0,T;B)$ = (strongly) measurable B-valued functions).

$$u_n \to u \text{ in measure iff}$$

$$\lim_{n\uparrow+\infty} \Big| \{t \in (0,T) : \|u_n(t) - u(t)\|_B \geq \sigma\} \Big| = 0 \quad \forall \sigma > 0,$$

(| $\cdot$ | denotes the Lebesgue measure).

- $\{u_n\} \subset L^p(0,T;B)$ is **p-uniformly integrable** iff

$$\lim_{|J|\downarrow 0} \sup_{n\in\mathbb{N}} \int_J \|u_n(t)\|_B^p \, dt = 0.$$

FROM L^p COMPACTNESS
TO COMPACTNESS IN MEASURE

Fact 1: Let $\{u_n\}$ fulfil the **integral equicontinuity condition.**
Then,

$\{u_n\}$ is **p-uniformly integrable.**

Fact 2: Let $\{u_n\} \subset L^p(0,T;B)$ be **p-uniformly integrable.**
Then,

$\{u_n\}$ is relatively compact **in $\boldsymbol{L^p(0,T;B)}$ iff**
$\{u_n\}$ is relatively compact **in measure**.

Supposing (IEC), we **turn to compactness in measure!!**

PLAN OF THE TALK

- ♢ Generalization of the **Aubin-Lions compactness in space** condition $\rightsquigarrow$ the **tightness** condition $\rightsquigarrow$ adopt a **probabilistic point of view**
- ♢ the **Young measures** approach
- ♢ the fundamental **compactness result** of **Young measures** theory
- ♢ **criterion** for compactness in **measure**
- ♢ **criterion** for L^p **compactness**

THE TIGHTNESS CONDITION

space compactness condition (SC1) $\rightsquigarrow$ **tightness condition**

$$\sup_{n\in\mathbb{N}} \int_0^T \mathscr{F}(t, u_n(t))\, dt < +\infty,$$

with $\mathscr{F} : (0,T)\times B \to [0,+\infty]$ measurable, s.t. for a.e. $t \in (0,T)$

$v \mapsto \mathscr{F}_t(v) := \mathscr{F}(t,v)$ is **l.s.c.**

$\{v \in B : \mathscr{F}_t(v) \le c\}$ are **compact** for any $c \ge 0$

i.e., $\mathscr{F}$ is a **normal coercive integrand** on $(0,T)\times B$.

Example: **the Aubin-Lions condition.** $X \subset B$ **compactly**,

$$\mathscr{F}(t,v) := \begin{cases} \|v\|_X^p & v \in X, \\ +\infty & \text{otherwise}, \end{cases} \qquad 1 \le p < \infty.$$

PARAMETRIZED (YOUNG) MEASURES

Definition A parametrized (Young) measure $\boldsymbol{\nu} := \{\nu_t\}_{t\in(0,T)}$ is family of **probability measures** ν_t on B s.t.

$$t \in (0,T) \mapsto \int_B \phi(\xi)\, d\nu_t(\xi) \quad \text{is} \quad \text{measurable} \quad \forall \phi \in C^b(B),$$

$C^b(B)$ the space of continuous and bounded functions on B.
We denote by $\mathcal{Y}(0,T;B)$ the set of Young measures.

Fubini's Theorem: For $\boldsymbol{\nu} = \{\nu_t\}_{t\in(0,T)} \in \mathcal{Y}(0,T;B)$ $\exists!$ measure ν on $(0,T)\times B$ s.t. for every measurable $\varphi : (0,T)\times B \to [0,+\infty]$

$$t \in (0,T) \mapsto \int_B \varphi(t,\xi) d\nu_t(\xi) \quad \text{is measurable},$$

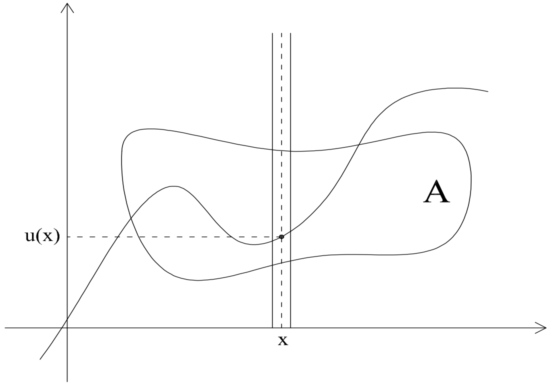
$$\int_{(0,T)\times B} \varphi(t,\xi)\, d\nu(t,\xi) = \int_0^T \Big(\int_B \varphi(t,\xi) d\nu_t(\xi) \Big)\, dt.$$

YOUNG MEASURE ASSOCIATED TO A FUNCTION

To $u \in \mathcal{M}(0,T;B)$ we associate a Young measure $\boldsymbol{\nu} = \{\nu_t\}_{t\in(0,T)}$ by $\nu_t := \delta_{u(t)}$ for a.e. $t \in (0,T)$, i.e.,

$$\nu(A) = \int_0^T \delta_{u(t)}(A_t)dt\,\forall A \subset (0,T)\times B,\ A_t := \{\xi \in B\ :\ (t,\xi) \in A\}.$$

ν is the **measure carried by the graph** of u .



Conversely, $\boldsymbol{\nu} \in \mathcal{Y}(0,T;B)$ is **associated** to a **function** if

the **support** of ν_t is a **singleton** for a.e. $t \in (0,T)$.

NARROW CONVERGENCE OF YOUNG MEASURES

Definition Let $\{\boldsymbol{\nu}_n\}$, $\boldsymbol{\nu} \in \mathcal{Y}(0,T;B)$: $\boldsymbol{\nu}_n$ **narrowly converges to** $\boldsymbol{\nu}$ $(\boldsymbol{\nu} \Rightarrow \boldsymbol{\nu})$ iff $\forall \varphi \in C^b((0,T) \times B)$

$$\lim_{n\uparrow+\infty} \int_{(0,T)\times B} \varphi(t,\xi) d\nu^n(t,\xi) = \int_{(0,T)\times B} \varphi(t,\xi) d\nu(t,\xi).$$

Link with the convergence in measure: Let $\{u_n\}$, $u \in \mathcal{M}(0,T;B)$ and $\{\boldsymbol{\nu}_n\}$, $\boldsymbol{\nu} \in \mathcal{Y}(0,T;B)$, with $\nu_t^n = \delta_{u_n(t)}$, $\nu_t = \delta_{u(t)}$ for a.e. $t \in (0,T)$. Then,

$$u_n \longrightarrow u \quad \textbf{in measure} \quad \Leftrightarrow \quad \boldsymbol{\nu}_n \Rightarrow \boldsymbol{\nu} \quad \textbf{narrowly}$$

Crucial Fact: If $u_n \leftrightarrow \boldsymbol{\nu}_n = \{\delta_{u_n(t)}\}_{t\in(0,T)}$ and $u_n \Rightarrow \boldsymbol{\mu}$ (i.e., $\boldsymbol{\nu}_n \Rightarrow \boldsymbol{\mu}$), then

$\{u_n\}$ **converges in measure iff** μ_t is a **Dirac mass** for a.e. $t \in (0,T)$.

COMPACTNESS FOR YOUNG MEASURES

Theorem [Balder, 1984]

Let $\{u_n\} \in \mathcal{M}(0,T;B)$ be
a **tight** sequence
w.r.t. a normal coercive integrand $\mathscr{F}$.
Then, there exists a **subsequence** u_{n_k} and
a **parametrized measure** $\boldsymbol{\mu} = \{\mu_t\}_{t\in(0,T)} \in \mathcal{Y}(0,T;B)$,
such that

$$u_{n_k} \Rightarrow \boldsymbol{\mu} \quad \text{as } k \uparrow \infty.$$

BACK TO COMPACTNESS IN MEASURE

- Replace the Aubin-Lions **space compactness condition** by the **tightness condition** (for a normal coercive integrand $\mathscr{F}$)

$$\sup_{n\in\mathbb{N}} \int_0^T \mathscr{F}(t, u_n(t))\, dt < +\infty.$$

- By BALDER'S theorem,

$$\exists\ \boldsymbol{\mu} = \{\mu_t\}_{t\in(0,T)} \in \mathcal{Y}(0,T;B) \quad \text{s.t.} \quad u_{n_k} \Rightarrow \boldsymbol{\mu} \quad \text{as } k \uparrow \infty.$$

- It is **sufficient** to show that μ_t is **concentrated**, i.e.

$$\mu_t = \delta_{u(t)} \quad \text{for a.e. } t \in (0,T),$$

in order to **conclude**

$$u_{n_k} \to u \quad \textbf{in measure} \text{ as } k \uparrow \infty.$$

INTEGRAL EQUICONTINUITY ⇒ CONCENTRATION

Technical passage: If

$$\lim_{h\downarrow 0} \sup_{n\in\mathbb{N}} \int_0^{T-h} \|u_n(t+h) - u_n(t)\|_B^p \, dt = 0,$$

then, **passing to the limit with Young measures,**

$$\int_0^T \Big(\iint_{B\times B} \|v-w\|_B \, d\mu_t(v)\, d\mu_t(w) \Big) dt = 0,$$

whence

$$\|v-w\|_B = 0 \quad \text{for } \mu_t \otimes \mu_t\text{-a.e. } (v,w) \in B\times B.$$

$\Rightarrow$ μ_t is a **Dirac mass** for a.e. $t \in (0,T)$.

A NEW CONCENTRATION CONDITION

It is possible to replace (IEC) by the

WEAK INTEGRAL EQUICONTINUITY CONDITION

$$\lim_{h\downarrow 0} \sup_{n\in\mathbb{N}} \int_0^{T-h} g(u_n(t+h), u_n(t))\, dt = 0,$$

$g : B \times B \to [0, +\infty]$, is **lower semicontinuous** and (LSC)

g is **compatible with $\mathscr{F}$** (of the tightness cond.), i.e.

$$u, v \in D(\mathscr{F}_t),\ g(u, v) = 0 \quad \Rightarrow \quad u = v \qquad \text{(COMP)}$$

with $D(\mathscr{F}_t) := \{v \in B : \mathscr{F}(t, v) < +\infty\}$ for a.e. $t \in (0, T)$.

Metric extension of (IEC): $g(u, v) := d_B(u, v)\ \forall (u, v) \in B \times B$.

WEAK INTEGRAL EQUICONTINUITY

Let us gain **further insight** into

$$\lim_{h\downarrow 0}\sup_{n\in\mathbb{N}}\int_0^{T-h} g(u_n(t+h),u_n(t))\,dt=0. \qquad \text{(WIEC)}$$

- The **lower-semicontinuity** (LSC) of g allows to

pass to the limit with Young measures and obtain

$$\int_0^T \Big(\iint_{B\times B} g(v,w)\,d\mu_t(v)\,d\mu_t(w)\Big)=0.$$

- Thanks to the **compatibility condition** (COMP),

$$g(v,w)=0 \quad \text{for } \mu_t\otimes\mu_t\text{-a.e.}(v,w)\in B\times B$$

$\Rightarrow$ μ_t is a **Dirac mass** for a.e. $t\in(0,T)$.

OUR CRITERION FOR COMPACTNESS IN MEASURE

Theorem 1 “Compactness in measure=tightness + W.I.E.C.” [R., Savaré (Ann. Sc. Norm. Sup. Pisa Cl. Sci. 2003)]
Sufficiency: Let $\{u_n\} \subset \mathcal{M}(0,T;B)$ such that

$$\{u_n\} \text{ is } \textbf{tight},$$
$$\{u_n\} \text{ fulfils (WIEC)}.$$

Then,

$$\{u_n\} \text{ is relatively compact in } \mathcal{M}(0,T;B).$$

Necessity: Conversely, if $\{u_n\}$ is relatively compact in measure, then $\{u_n\}$ is tight w.r.t. a normal coercive integrand $\mathscr{F}$ independent of the variable t and (WIEC) holds for any bounded continuous (semi-)distance g on B.

OUR CRITERION FOR L^p COMPACTNESS

Theorem 2 “L^p compactness=integral equicontinuity +tightness” [R., Savaré (Ann. Sc. Norm. Sup. Pisa Cl. Sci. 2003)]

A bounded sequence $\{u_n\}$ in $L^p(0,T;B)$
is **relatively compact iff**

$\{u_n\}$ is **tight** w.r.t. a normal coercive integrand $\mathscr{F}$ and

$$\lim_{h\downarrow 0}\sup_{n\in\mathbb{N}}\int_0^{T-h}\|u_n(t+h)-u_n(t)\|_B^p\,dt=0.$$

Converse of the Aubin-Lions theorem. If $\{u_n\}$ is relatively compact in $L^p(0,T;B)$, there exists $X\subset B$ **compactly** s. t.

$$\sup_{n\in\mathbb{N}}\int_0^T\|u_n(t)\|_X^p dt<+\infty.$$

REMARKS

- **Extension** of Thms. 1 and 2 to B-valued functions defined on a **bounded domain** $\Omega \subset \mathbb{R}^d$, $d \geq 1$.
- Theorem 1 is of **metric nature** and still holds if

$$(B, \|\cdot\|_B) \rightsquigarrow (B, d_B).$$

- **Extension** of Theorem 1 to **weak topologies**: $\{u_n\} \in \mathcal{M}(0,T;B)$ weakly converges in measure to u iff

$${}_{B'}\langle f, u_n \rangle_B \to {}_{B'}\langle f, u \rangle_B \quad \text{in measure} \quad \forall f \in B'.$$

- Allowing for a **time-dependent** integrand $\mathscr{F}$ in the tightness condition opens to non-trivial applications of Theorems 1 and 2 to **evolutionary** PDEs (e.g., extend a compactness result of LUCKHAUS, VISINTIN for the **Stefan Problem** with the Gibbs-Thomson law).