A vanishing viscosity approach to rate-independent modelling in metric spaces

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joint work (in progress) with
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Rate-independence, Homogenization and Multiscaling

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The subdifferential formulation of rate-independent problems

Rate-independent systems can be modelled by

$$\partial\Psi(u'(t)) + \partial_u\mathcal{E}(t,u(t)) \ni 0 \quad \text{in } B', \qquad t \in (0,T) \tag{DNE}$$

- B Banach space;
- $\Psi : B \to [0,+\infty]$, $\Psi(0) = 0$, l.s.c. and **convex**
- ∂ subdifferential in the sense of **convex analysis**
- Ψ **positively 1-homogeneous** $\leftrightarrow$ (DNE) invariant under time rescaling
- $\mathcal{E} : [0,T] \times B \to (-\infty,+\infty]$ **smooth** w.r.t. $t \in (0,T)$
- ∂_u "subdifferential" of $\mathcal{E}$ **w.r.t. the second variable**

Drawbacks

The subdifferential formulation works well in a smooth/convex setting.
Standard regularity of solutions is $u \in \mathrm{BV}(0,T;B)$ (u may have **jumps!!!**):
how to handle derivatives?

Looking for a derivative-free formulation

Applications [... most of the people here....]

1. **quasistatic solid-solid phase transformations (in SMA)**
2. **linearized & finite-strain elastoplasticity**
3. **quasistatic propagation of fractures**
4. **damage**
5. **delamination problems**
6. **ferromagnetism, ferroelectricity**
7. **shape evolution of debonding membranes**......

In applications

- $\mathcal{E}$ may be **neither smooth nor convex!**
- B may be **non reflexive** (e.g. L^1 in SMA)
- B may **lack a linear structure** (e.g. in fractures)

Need of a **derivative-free** formulation (without u' and $\partial_u \mathcal{E}$!)

The energetic formulation

In this spirit: generalized formulations for doubly nonlinear problems based on **global variational principles**: **[Visintin'01], [Mielke-Ortiz07], [Stefanelli06-07]**

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Energetic solutions [Mielke-Theil'99,'04], [Mielke-Theil-Levitas'02]

$u : [0, T] \to B$ satisfying **global stability condition** & **energy balance**

$$\mathcal{E}(t, u(t)) \leq \mathcal{E}(t, z) + \mathcal{D}(u(t), z) \qquad \forall z \in B,$$

$$\mathcal{E}(t, u(t)) + \mathrm{Diss}_{\mathcal{D}}(u, [0, t]) = \mathcal{E}(t, u(0)) + \int_0^t \partial_t \mathcal{E}(r, u(r))\,\mathrm{d}r.$$

Pro's

✓ Completely **derivative-free** $\rightsquigarrow$ adaptable to more general ambient spaces (general topological spaces **[Mainik-Mielke'05], [Bucur-Buttazzo'07] (Minimizing movements approach)**)

✓ **equivalence** with the **subdifferential** formulation (DNE) if $\mathcal{E}$ **convex**

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BUT (in the **non convex** case), **global stability** forces energetic solutions to **jump early** to avoid energy losses (Maxwell rule)

Viscous regularizations & jumps & local, rather than global, stability

Aims

- **model** ("natural") **jumps** (due to $u \in \mathrm{BV}(0, T; B)$)
- obtain **solutions jumping later** (than via the Maxwell rule)

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Consider solutions arising as limits of viscous regularizations for **vanishing viscosity**: selection criterion for mechanically feasible jumps

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Vanishing viscosity in the applications

- quasistatic evolution of fractures: [Toader-Zanini'06], [Cagnetti'07], [Cagnetti-Toader'07], [Knees-Mielke-Zanini'07], leading to **local stability**-oriented formulations: [Dal Maso-Toader'02], [Negri-Ortner'07], [Garroni-Larsen07] **(threshold evolutions in damage)**..
- plasticity with softening: [Dal Maso-DeSimone-Mora-Morini'06]
- Kurzweil formulation of rate-independent processes with **convex energies & discontinuous inputs**: [Krejčí-Liero'07]

The vanishing viscosity analysis by Efendiev & Mielke

Problem

In the vanishing viscosity limit:

- local stability
- energy inequality

may not be enough for controlling jumps. ¿ Which further conditions better describe them?

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The approach by Efendiev-Mielke

- Jumps in the vanishing viscosity limit correspond to **viscous transitions** between stable states
- To capture the viscous transition path: **NOT SHRINK** jumps at a point, look at curves with their **arc length parametrization**
- Asymptotic analysis of (reparametrized) trajectories in an extended phase space

The vanishing viscosity analysis by Efendiev & Mielke

[Efendiev-Mielke, J. Convex Anal.'06]

Setting:

- B **finite dimensional** space
- $\mathcal{E} \in \mathrm{C}^1([0,T] \times B; \mathbb{R}^+)$
- $C_1\|u\| \leq \Psi(u) \leq C_2\|u\| \quad \forall\, u \in B$

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The **viscous regularization** of Ψ:

$$\Phi_\varepsilon(u) := \Psi(u) + \frac{\varepsilon}{2}\|u\|^2 \quad \forall\, \varepsilon > 0.$$

Let $\{u_\varepsilon\}_{\varepsilon>0}$ be the family of solutions of the Cauchy problem

$$\begin{cases} \partial\Phi_\varepsilon(u_\varepsilon{}'(t)) + \mathrm{D}\mathcal{E}(t, u_\varepsilon(t)) \ni 0 \quad t \in (0,T), \\ u_\varepsilon(0) = u_0. \end{cases}$$

Problem: limit behaviour of $\{u_\varepsilon\}$ as $\varepsilon \searrow 0$

A rescaling technique

- **Arc length parametrization** of the graph $\{(t, u_\varepsilon(t)) \,:\, t \in [0, T]\}$:

$$s_\varepsilon(t) := t + \int_0^t \|u_\varepsilon{}'(s)\| \mathrm{d}s$$

$\{s_\varepsilon\}_\varepsilon$ is bounded in $L^\infty(0, T)$: up to a subseq. $s_\varepsilon(T) \to \widehat{T}$.

- Introduce the rescaled functions

$$\widehat{t}_\varepsilon(s) := s_\varepsilon^{-1}(s), \quad \widehat{u}_\varepsilon(s) := u_\varepsilon(\widehat{t}_\varepsilon(s)) \quad \forall s \in [0, s_\varepsilon(T)]$$

- From the **normalization condition**

$$\widehat{t}_\varepsilon'(s) + \|\widehat{u}_\varepsilon'(s)\| = 1 \quad \text{for a.e. } s \in (0, s_\varepsilon(T))$$

$\Rightarrow$ a priori estimates for $\{\widehat{t}_\varepsilon\}$, $\{\widehat{u}_\varepsilon\}$

- Ascoli-Arzelà + finite dimension

$$\widehat{t}_\varepsilon \to \widehat{t}, \quad \widehat{u}_\varepsilon \to \widehat{u} \quad \text{uniformly on } [0, \widehat{T}]$$

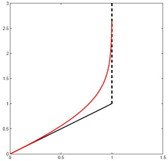
A rescaling technique

The **limit problem** solved by $(\widehat{t}, \widehat{u})$

$$\begin{cases} \partial\widehat{\Psi}(\widehat{u}'(s)) + \mathrm{D}\mathcal{E}(\widehat{t}(s), \widehat{u}(s)) \ni 0 \quad s \in (0, \widehat{T}) \\ \widehat{u}(0) = u_0, \quad \widehat{t}(0) = 0, \quad \widehat{t}(\widehat{T}) = T, \\ \widehat{t}'(s) + \|\widehat{u}'(s)\| = 1 \quad s \in (0, \widehat{T}) \end{cases}$$

where

$$\widehat{\Psi}(u') := \begin{cases} \Psi(u') & \|u'\| \leq 1, \\ +\infty & \|u'\| > 1 \end{cases}$$



Vanishing viscosity limit: dry friction vs. viscous slips

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$\widehat{\Psi}$ is NOT 1-homogeneous $\Rightarrow$ the problem is NOT rate-independent!

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"Dry friction vs. viscous slips"

Three regimes

$\|\widehat{u}'(s)\| = 0 \Leftrightarrow \widehat{t}'(s) = 1$ — Sticking

$\|\widehat{u}'(s)\| < 1 \Leftrightarrow \widehat{t}'(s) \in (0,1)$ — Dry friction motion

$\|\widehat{u}'(s)\| = 1 \Leftrightarrow \widehat{t}'(s) = 0$ — Viscous slip

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"Dry friction vs. viscous slips"

Three regimes

1. for $\|\widehat{u}'(s)\| = 0$ the system is stationary
2. for $\|\widehat{u}'(s)\| < 1$ the system is driven by rate-independent dissipation ($\sim$ dry friction): reparametrizing $\widehat{u}$ leads to a standard rate-independent problem
3. $\|\widehat{u}'(s)\| = 1$ corresponds to viscous transition between stable states ("instantaneous" w.r.t. the slow time scale, whence $\widehat{t}'(s) = 0$); viscous path described by a gradient flow

Towards metric spaces

In applications

- $\mathcal{E}$ may be **neither smooth nor convex!**
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- B may **lack a linear structure** (e.g. in fractures)

$\Rightarrow$ extend the Efendiev-Mielke analysis to a **metric setting**

The metric framework will lead to local, rather than global, stability!

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Outline

In a **metric framework**:

1. **Approximate** rate-independent evolutions with **viscous evolutions** **[Mielke, R., Savaré, work in progress]**
2. Analysis of doubly nonlinear evolution equations where **dissipation with superlinear growth**: existence & approximation of solutions **[Mielke, R., Savaré, preprint'07]**

Doubly nonlinear evolutions in metric spaces

Analysis of

$$\partial\Psi(u'(t)) + \partial_u\mathcal{E}(t,u(t)) \ni 0 \quad t \in (0,T) \tag{DNE}$$

Ψ with superlinear growth

in the framework of a **metric space (X,d)**.

Relying on: theory of **gradient flows** in metric spaces (i.e. **quadratic** Ψ):

- **De Giorgi, Marino, Saccon, Tosques, Degiovanni, Ambrosio** '80 $\sim$ '90 $\rightsquigarrow$ theory of **Curves of Maximal Slope** and **Minimizing Movements**
- [*Gradient flows in metric spaces*, **Ambrosio-Gigli-Savaré** 2005] $\rightsquigarrow$ systematic theory of existence, approximation & uniqueness of solutions of metric gradient flows, with applications to gradient flows in Wasserstein spaces.

Towards the metric formulation

Problem:

How to formulate

$$\text{“ } \partial\Psi(u'(t)) + \partial_u\mathcal{E}(t,u(t)) = 0, \quad t \in (0,T) \text{ ”}$$

without **a linear/differential structure** on X?

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Heuristics:
If the **chain rule** holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,u(t)) - \partial_t\mathcal{E}(t,u(t)) = \langle \partial_u\mathcal{E}(t,u(t)), u'(t)\rangle$$

then (DNE) is equivalent to

$$\Psi(u'(t))) + \Psi^*(-\partial_u\mathcal{E}(t,u(t))) + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,u(t)) - \partial_t\mathcal{E}(t,u(t)) = 0 \quad t \in (0,T)$$

(abuse of notation: $\partial_u\mathcal{E}(t,u(t)) \sim$ singleton...)

Towards the metric formulation

In the particular case

$$\Psi(x) := \frac{|x|^p}{p}, \quad 1 < p < \infty, \qquad \Psi^*(x) := \frac{|x|^q}{q}, \quad \frac{1}{p} + \frac{1}{q} = 1$$

$$\Psi(u'(t)) + \Psi^*(-\partial_u \mathcal{E}(t, u(t))) + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t, u(t)) - \partial_t \mathcal{E}(t, u(t)) = 0 \quad t \in (0, T)$$

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New formulation features the **modulus of derivatives,** rather than derivatives!

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Adaptable to metric spaces upon introducing suitable **"metric surrogates"** of **"modulus of derivatives"**.

The metric derivative

- **Setting:** (X, d) complete metric space

Metric derivative

- We say that a curve $u : [0, T] \to X$ is **absolutely continuous** if

$$\exists m \in L^1(0,T) \ : \quad d(u(t),u(s)) \leq \int_s^t m(r)\,\mathrm{d}r \quad \forall\, 0 \leq s \leq t \leq T.$$

- Given $u \in \mathrm{AC}(0,T;X)$, its **metric derivative**

$$|u'|(t) := \lim_{h \to 0} \frac{d(u(t),u(t+h))}{|h|} \quad \text{for a.e. } t \in (0,T)$$

$$\|u'(t)\| \rightsquigarrow |u'|(t)$$

Slope & Chain rule

- **Setting:** (X, d) complete metric space

Local slope & Chain rule

- Given $\mathcal{E} : [0, T] \times X \to (-\infty, +\infty]$ and $u \in D(\mathcal{E}(t, \cdot))$, the **local slope** of $\mathcal{E}(t, \cdot)$ at u is

$$|\partial\mathcal{E}|(t, u) := \limsup_{v \to u} \frac{(\mathcal{E}(t, u) - \mathcal{E}(t, v))^+}{d(u, v)}$$

$$\| -\partial_u \mathcal{E}(t, u)\| \rightsquigarrow |\partial\mathcal{E}|(t, u)$$

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$$\| - \partial_u \mathcal{E}(t, u)\| \rightsquigarrow |\partial \mathcal{E}|(t, u)$$

- $\mathcal{E}$ complies with the **chain rule** w.r.t. $|\partial \mathcal{E}|$ if $\forall v \in \mathrm{AC}(0, T; D(\mathcal{E}))$ the map $t \mapsto \mathcal{E}(t, v(t))$ is **absolutely continuous** and

$$\partial_t \mathcal{E}(t, v(t)) - \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}(t, v(t)) \leq |v'|(t)\, |\partial \mathcal{E}|(t, v(t)) \quad \text{for a.e. } t \in (0, T).$$

The metric formulation

- **Basic setting**:
  - (X, d) complete metric space
  - **Energy** $\rightsquigarrow$ $\mathcal{E} : [0, T] \times X \to (-\infty, +\infty]$ l.s.c., complying with the **chain rule** w.r.t. $|\partial\mathcal{E}|$
  - **Dissipation** $\rightsquigarrow$ $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ l.s.c., convex, $\psi(0) = 0$, with

$$\lim_{x \to +\infty} \frac{\psi(x)}{x} = +\infty$$

Metric formulation

A curve $u \in \mathrm{AC}(0, T; X)$ satisfies the **metric formulation** of

$$\text{“}\partial\Psi(u'(t)) + \partial_u\mathcal{E}(t, u(t)) = 0 \quad t \in (0, T)\text{”}$$

if for a.e. $t \in (0, T)$

$$\psi(|u'|(t)) + \psi^*(|\partial\mathcal{E}|(t, u(t))) + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t, u(t)) - \partial_t\mathcal{E}(t, u(t)) = 0$$

An existence result

Theorem [Mielke, R., Savaré, preprint'07]

- $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ convex, l.s.c., $\psi(0) = 0$, superlinear growth
- $\mathcal{E} : [0, T] \times X \to (-\infty, +\infty]$ smooth w.r.t. $t \in [0, T]$
- $\mathcal{E}$ l.s.c. and coercive w.r.t. $u \in X$, **chain rule** w.r.t. $|\partial\mathcal{E}|$
- $u \mapsto |\partial\mathcal{E}|(t, u)$ is l.s.c. (along bounded energy sequences)

Then, for all $u_0 \in D(\mathcal{E})$ **there exists** a curve $u \in \mathrm{AC}(0, T; X)$ such that $u(0) = u_0$ and

$$\psi(|u'|(t)) + \psi^* \left(|\partial\mathcal{E}|(t, u(t))\right) = \partial_t\mathcal{E}(t, u(t)) - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t, u(t)) \quad \text{for a.e. } t \in (0, T)\,.$$

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Proof: approximation via time discretization (**incremental minimization** problems), based on De Giorgi's **Minimizing Movements** theory.
Applications: existence results for doubly nonlinear evolution equations in (possibly non reflexive) spaces

Approximation of rate-independent problems with viscous evolutions

Second step

In the metric space (X, d), approximate

$$\partial\Psi(u'(t)) + \partial_u\mathcal{E}(t, u(t)) \ni 0 \quad \text{in } B' \qquad t \in (0, T), \tag{DNE}$$

Ψ **1-positively homogeneous**

with the viscous evolution

$$\varepsilon u'(t) + \partial\Psi(u'(t)) + \partial_u\mathcal{E}(t, u(t)) \ni 0 \quad t \in (0, T), \qquad \text{as } \varepsilon \searrow 0$$

Approximation of rate-independent problems with viscous evolutions

In the metric setting

- (X, d) metric space
- $\mathcal{E} : [0, T] \times X \to \mathbb{R} \cup \{+\infty\}$: assumptions for $\exists$ + **Chain rule**
- $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ convex **1-positively homogeneous** ($\psi(r) = r \;\; \forall\, r \in \mathbb{R}^+$)
- Viscous regularization of ψ: $\psi_\varepsilon(x) := x + \frac{\varepsilon}{2}x^2 \;\forall\, x \geq 0 \;\forall\, \varepsilon > 0$.
- $\{u_\varepsilon\}_{\varepsilon>0} \subset \mathrm{AC}(0, T; X)$: **metric solutions** of

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t, u_\varepsilon(t)) - \partial_t\mathcal{E}(t, u_\varepsilon(t)) = \\ \qquad - \psi_\varepsilon(|u_\varepsilon{}'|(t)) - \psi_\varepsilon{}^*(|\partial\mathcal{E}|\,(t, u_\varepsilon(t))) \;\; \text{for a.e. } t \in (0, T) \\ u_\varepsilon(0) = u_0. \end{cases}$$

- **Problema:** ¿ limit of $\{u_\varepsilon\}$ as $\varepsilon \searrow 0$?

Vanishing viscosity revisited

Extend the Mielke-Efendiev technique to the metric setting:

♣ reparametrize by the **arc length** of $\{(t, u_\varepsilon(t)) : t \in [0, T]\}$:

$$\begin{cases} s_\varepsilon(t) := t + \int_0^t |u_\varepsilon{}'|(r)\,\mathrm{d}r \\ \widehat{t}_\varepsilon(s) := s_\varepsilon^{-1}(s), \quad \widehat{u}_\varepsilon(s) := u_\varepsilon(\widehat{t}_\varepsilon(s)) \quad s \in [0, s_\varepsilon(T)] \end{cases}$$

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♣ you pass from

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t, u_\varepsilon(t)) - \partial_t \mathcal{E}(t, u_\varepsilon(t)) = \\ \qquad - \psi_\varepsilon(|u_\varepsilon{}'|(t)) - \psi_\varepsilon{}^*(|\partial\mathcal{E}|(t, u_\varepsilon(t))) & t \in (0,T) \\ u_\varepsilon(0) = u_0. \end{cases}$$

Vanishing viscosity revisited

Extend the Mielke-Efendiev technique to the metric setting:

♣ reparametrize by the **arc length** of $\{(t, u_\varepsilon(t)) : t \in [0,T]\}$:

$$\begin{cases} s_\varepsilon(t) := t + \int_0^t |u_\varepsilon{}'|(r)\,\mathrm{d}r \\ \widehat{t}_\varepsilon(s) := s_\varepsilon^{-1}(s), \quad \widehat{u}_\varepsilon(s) := u_\varepsilon(\widehat{t}_\varepsilon(s)) \quad s \in [0, s_\varepsilon(T)] \end{cases}$$

♣ to

$$\begin{cases} \widehat{t}_\varepsilon(0) = 0 \qquad \widehat{t}_\varepsilon(s_\varepsilon(T)) = T \\ \widehat{t}_\varepsilon'(s) + |\widehat{u}_\varepsilon'|(s) = 1 \quad \text{for a.e. } s \in (0, s_\varepsilon(T)) \\ \text{rescaled metric formulation of (DNE) } (\psi_\varepsilon, \mathcal{E}) \end{cases}$$

Problem: ¿ asymptotic analysis of $\{(\widehat{t}_\varepsilon, \widehat{u}_\varepsilon)\}$ as $\varepsilon \searrow 0$?

The asymptotic analysis result

Let

$$\widehat{\psi}(r) := \begin{cases} r & r \in [0,1], \\ +\infty & r > 1, \end{cases} \qquad \widehat{T} := \lim_{\varepsilon \downarrow 0} s_\varepsilon(T).$$

Theorem [Mielke, R., Savaré, in preparation '07]

Assumptions: like for ∃ of metric solution (in particular, **chain rule**).

Then, up to a subsequence, $\{(\widehat{t}_\varepsilon, \widehat{u}_\varepsilon)\}$ converges as $\varepsilon \searrow 0$ to $(\widehat{t}, \widehat{u}) \in \mathrm{C}^0_{\mathrm{Lip}}([0, \widehat{T}]; [0, T] \times X)$, which satisfies

$$\begin{cases} \widehat{t}(0) = 0 \qquad \widehat{t}(\widehat{T}) = T \\ \widehat{t}'(s) + |\widehat{u}'|(s) = 1 \quad \text{for a.e. } s \in (0, \widehat{T}) \end{cases}$$

and the **"rescaled metric formulation"**

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\widehat{t}(s), \widehat{u}(s)) - \partial_t \mathcal{E}(\widehat{t}(s), \widehat{u}(s))\, \widehat{t}'(s) = -\widehat{\psi}(|\widehat{u}'|(s)) - \widehat{\psi}^* \left(|\partial \mathcal{E}|\, (\widehat{t}(s), \widehat{u}(s))\right) \quad s \in (0, \widehat{T}).$$

More insight into the vanishing viscosity limit

$$(\widehat{t},\widehat{u}) \in \mathrm{C}^0_{\mathrm{Lip}}([0,\widehat{T}];[0,T]\times X) \qquad \widehat{t}(0)=0 \qquad \widehat{t}(\widehat{T})=T$$

$$\widehat{t}'(s)+\widehat{u}'(s)=1 \quad \text{for a.e. } s\in(0,\widehat{T})$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\widehat{t}(s),\widehat{u}(s)) - \partial_t\mathcal{E}(\widehat{t}(s),\widehat{u}(s))\,\widehat{t}'(s) = -\widehat{\psi}(|\widehat{u}'|(s)) - \widehat{\psi}^*\left(|\partial\mathcal{E}|\left(\widehat{t}(s),\widehat{u}(s)\right)\right) \quad \text{for a.e. } s\in(0,\widehat{T})$$

More insight into the vanishing viscosity limit

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$$\begin{cases} \textbf{En. id.:} & \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\widehat{t}(s),\widehat{u}(s)) - \partial_t\mathcal{E}(\widehat{t}(s),\widehat{u}(s))\,\widehat{t}'(s) = -|\widehat{u}'|(s)\,|\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s)) \\ \textbf{Constraint:} & |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s)) \in \partial\widehat{\psi}(|\widehat{u}'|(s)) \ \text{ for a.e. } s\in(0,\widehat{T}) \end{cases}$$

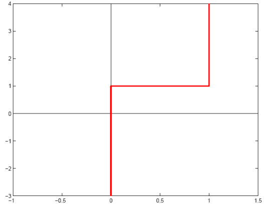
More insight into the vanishing viscosity limit

$$(\widehat{t},\widehat{u}) \in \mathrm{C}^0_{\mathrm{Lip}}([0,\widehat{T}];[0,T]\times X) \qquad \widehat{t}(0)=0 \qquad \widehat{t}(\widehat{T})=T$$

$$\widehat{t}'(s)+\widehat{u}'(s)=1 \quad \text{for a.e. } s\in(0,\widehat{T})$$

$$\begin{cases} \textbf{En. id.:} & \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\widehat{t}(s),\widehat{u}(s))-\partial_t\mathcal{E}(\widehat{t}(s),\widehat{u}(s))\,\widehat{t}'(s)=-|\widehat{u}'|(s)\,|\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s)) \\ \textbf{Constraint:} & |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s))\in\partial\widehat{\psi}(|\widehat{u}'|(s)) \quad \text{for a.e. } s\in(0,\widehat{T}) \end{cases}$$

$$|\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s))\in\partial\widehat{\psi}(|\widehat{u}'|(s)) \quad \text{for a.e. } s\in(0,\widehat{T})$$



More insight into the vanishing viscosity limit

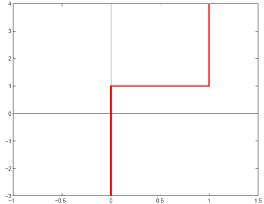
$$(\widehat{t},\widehat{u}) \in \mathrm{C}^0_{\mathrm{Lip}}([0,\widehat{T}];[0,T]\times X) \qquad \widehat{t}(0)=0 \qquad \widehat{t}(\widehat{T})=T$$

$$\widehat{t}'(s)+\widehat{u}'(s)=1 \quad \text{for a.e. } s\in(0,\widehat{T})$$

En. id.: $$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\widehat{t}(s),\widehat{u}(s)) - \partial_t\mathcal{E}(\widehat{t}(s),\widehat{u}(s))\,\widehat{t}'(s) = -|\widehat{u}'|(s)\;|\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s))$$

Three regimes: $$\begin{cases} |\widehat{u}'|(s)=1 \ (\Leftrightarrow \widehat{t}'(s)=0\,) & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s))\geq 1 \\ |\widehat{u}'|(s)\in(0,1) \ (\Leftrightarrow \widehat{t}'(s)\in(0,1)\,) & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s)) = 1 \\ |\widehat{u}'|(s)=0 \ (\Leftrightarrow \widehat{t}'(s)=1\,) & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s))\leq 1 \end{cases}$$

$$|\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s)) \in \partial\widehat{\psi}(|\widehat{u}'|(s)) \quad \text{for a.e. } s\in(0,\widehat{T})$$



Towards Metric Parametrized Rate-Independent Flows

These are the properties to retain:

$$(\widehat{t},\widehat{u}) \in \mathrm{AC}(0,\widehat{T};[0,T]\times X) \qquad \widehat{t}(0)=0 \qquad \widehat{t}(\widehat{T})=T$$

$$\widehat{t}'(s)+\widehat{u}'(s) > 0 \quad \text{for a.e. } s \in (0,\widehat{T})$$

En. id.: $\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\widehat{t}(s),\widehat{u}(s)) - \partial_t\mathcal{E}(\widehat{t}(s),\widehat{u}(s))\,\widehat{t}'(s) = -|\widehat{u}'|(s)\ |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s))$

Three regimes: $\begin{cases} \widehat{t}'(s)=0 & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s)) \geq 1 \\ \widehat{t}'(s)\,|\widehat{u}'|(s) > 0 & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s)) = 1 \\ |\widehat{u}'|(s)=0 & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s)) \leq 1 \end{cases}$

Metric Parametrized Rate-Independent Flows

Definition

A pair $(\widehat{t}, \widehat{u}) \in \mathrm{AC}(0, \widehat{T}; [0, T] \times X)$ is a **metric parametrized rate-independent flow** if

1. $\widehat{t}$ is non-decreasing, with $\widehat{t}(0) = 0$ and $\widehat{t}(\widehat{T}) = T$
2. there holds
$$\widehat{t}'(s) + \widehat{u}'(s) > 0 \quad \text{for a.e. } s \in (0, \widehat{T})$$
3. the map $s \in [0, \widehat{T}] \mapsto \mathcal{E}(\widehat{t}(s), \widehat{u}(s))$ is absolutely continuous and

En. id.: $$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\widehat{t}(s), \widehat{u}(s)) - \partial_t \mathcal{E}(\widehat{t}(s), \widehat{u}(s))\, \widehat{t}'(s) = -|\widehat{u}'|(s)\ |\partial\mathcal{E}|\left(\widehat{t}(s), \widehat{u}(s)\right)$$

Three regimes: $$\begin{cases} \widehat{t}'(s) = 0 & \Rightarrow\ |\partial\mathcal{E}|\left(\widehat{t}(s), \widehat{u}(s)\right) \geq 1 \\ \widehat{t}'(s)\,|\widehat{u}'|(s) > 0 & \Rightarrow\ |\partial\mathcal{E}|\left(\widehat{t}(s), \widehat{u}(s)\right) = 1 \\ |\widehat{u}'|(s) = 0 & \Rightarrow\ |\partial\mathcal{E}|\left(\widehat{t}(s), \widehat{u}(s)\right) \leq 1 \end{cases}$$

Metric Parametrized Rate-Independent Flows: rate-invariance

Metric Parametrized Rate-Independent Flow

En. id.: $\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\widehat{t}(s),\widehat{u}(s)) - \partial_t\mathcal{E}(\widehat{t}(s),\widehat{u}(s))\,\widehat{t}'(s) = -|\widehat{u}'|(s)\,|\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s))$

Three regimes: $\begin{cases} \widehat{t}'(s) = 0 & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s)) \geq 1 \\ \widehat{t}'(s)\,|\widehat{u}'|(s) > 0 & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s)) = 1 \\ |\widehat{u}'|(s) = 0 & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s)) \leq 1 \end{cases}$

- Approximable (via vanishing viscosity) solutions are MPRIFs.
- The class of MPRIF is **invariant** for (strictly increasing) reparametrizations $\Rightarrow$ MPRIF is a **truly rate-independent** notion.
- Notion **compatible with other vanishing viscosity approximations**, based on different reparametrizations (not necessarily by arc-length)!
- Existence can also be proved via time discretization (**Minimizing Movements**)

Metric Parametrized Rate-Independent Flows: flow regimes

Metric Parametrized Rate-Independent Flows

En. id.: $\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\widehat{t}(s),\widehat{u}(s)) - \partial_t\mathcal{E}(\widehat{t}(s),\widehat{u}(s))\,\widehat{t}'(s) = -|\widehat{u}'|(s)\;|\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s))$

Three regimes: $$\begin{cases} |\widehat{u}'|(s) = 0 & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s)) \leq 1 \\ \widehat{t}'(s)\,|\widehat{u}'|(s) > 0 & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s)) = 1 \\ \widehat{t}'(s) = 0 & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s)) \geq 1 \end{cases}$$

Mechanical interpretation

- **sticking** $\leftrightarrow$ $|\widehat{u}'|(s) = 0$
- **sliding** (dry friction motion) $\leftrightarrow$ $\widehat{t}'(s)|\widehat{u}'|(s) > 0$
- **viscous slip** $\leftrightarrow$ $\widehat{t}'(s) = 0.$

Flow regimes

Metric Parametrized Rate-Independent Flows

$$\widehat{t}'(s) \geq 0\,, \qquad \widehat{t}'(s) + \widehat{u}'(s) > 0 \quad \text{for a.e. } s \in (0, \widehat{T})$$

En. id:
$$\mathcal{E}(\widehat{t}(s_2), \widehat{u}(s_2)) - \mathcal{E}(\widehat{t}(s_1), \widehat{u}(s_1)) = \int_{s_1}^{s_2} \Big(\partial_t \mathcal{E}(\widehat{t}(s), \widehat{u}(s))\, \widehat{t}'(s) - |\widehat{u}'|(s)\, |\partial \mathcal{E}|\,(\widehat{t}(s), \widehat{u}(s)) \Big)\, \mathrm{d}s \quad \forall\, 0 \leq s_1 \leq s_2 \leq \widehat{T}$$

Differential conditions:
$$\begin{cases} |\widehat{u}'|(s) = 0 & \Rightarrow\ |\partial \mathcal{E}|\,(\widehat{t}(s), \widehat{u}(s)) \leq 1 \\ \widehat{t}'(s)\, |\widehat{u}'|(s) > 0 & \Rightarrow\ |\partial \mathcal{E}|\,(\widehat{t}(s), \widehat{u}(s)) = 1 \\ \widehat{t}'(s) = 0 & \Rightarrow\ |\partial \mathcal{E}|\,(\widehat{t}(s), \widehat{u}(s)) \geq 1 \end{cases}$$

Sticking: stationary set

$$\mathcal{S} := \Big\{ s_0 \in (0, \widehat{T}) : |\widehat{u}'|(s) = 0 \text{ in a neighb. of } s_0 \Big\}$$

- $|\widehat{u}'|(s) = 0 \Rightarrow \widehat{t}'(s) > 0$ and $|\partial \mathcal{E}|\,(\widehat{t}(s), \widehat{u}(s)) \leq 1$ (**local stability**)
- in a neighb. $I(s_0)$ we have $\widehat{u}(s) \equiv \widehat{u}(s_0)$ and the **energy identity**

$$\mathcal{E}(\widehat{t}(s_2), \widehat{u}(s_0)) - \mathcal{E}(\widehat{t}(s_1), \widehat{u}(s_0)) = \int_{s_1}^{s_2} \partial_t \mathcal{E}(\widehat{t}(s), \widehat{u}(s_0))\, \widehat{t}'(s)\, \mathrm{d}s \ \forall\, s_1 \leq s_2 \in I(s_0)\,.$$

Flow regimes

Metric Parametrized Rate-Independent Flows

$$\widehat{t}'(s) \geq 0, \qquad \widehat{t}'(s) + \widehat{u}'(s) > 0 \quad \text{for a.e. } s \in (0, \widehat{T})$$

En. id: $\mathcal{E}(\widehat{t}(s_2), \widehat{u}(s_2)) - \mathcal{E}(\widehat{t}(s_1), \widehat{u}(s_1)) = \int_{s_1}^{s_2} \Big(\partial_t \mathcal{E}(\widehat{t}(s), \widehat{u}(s))\, \widehat{t}'(s) - |\widehat{u}'|(s)\, |\partial\mathcal{E}|\,(\widehat{t}(s), \widehat{u}(s)) \Big)\, \mathrm{d}s \quad \forall\, 0 \leq s_1 \leq s_2 \leq \widehat{T}$

Differential conditions: $\begin{cases} |\widehat{u}'|(s) = 0 & \Rightarrow \ |\partial\mathcal{E}|\,(\widehat{t}(s), \widehat{u}(s)) \leq 1 \\ \widehat{t}'(s)\, |\widehat{u}'|(s) > 0 & \Rightarrow \ |\partial\mathcal{E}|\,(\widehat{t}(s), \widehat{u}(s)) = 1 \\ \widehat{t}'(s) = 0 & \Rightarrow \ |\partial\mathcal{E}|\,(\widehat{t}(s), \widehat{u}(s)) \geq 1 \end{cases}$

Sliding:

- $\widehat{t}'(s_0) > 0$ & $|\widehat{u}'|(s_0) > 0 \Rightarrow |\partial\mathcal{E}|\,(\widehat{t}(s_0), \widehat{u}(s_0)) = 1$ (**local stability**)
- in a neighb. $I(s_0)$ the **energy identity** reads $\forall\, s_1 \leq s_2 \in I(s_0)$

$$\mathcal{E}(\widehat{t}(s_2), \widehat{u}(s_2)) - \mathcal{E}(\widehat{t}(s_1), \widehat{u}(s_1)) = \int_{s_1}^{s_2} \Big(\partial_t \mathcal{E}(\widehat{t}(s), \widehat{u}(s))\, \widehat{t}'(s) - |\widehat{u}'|(s) \Big)\, \mathrm{d}s$$

Flow regimes

Metric Parametrized Rate-Independent Flows

$$\widehat{t}'(s) \geq 0\,, \qquad \widehat{t}'(s) + \widehat{u}'(s) > 0 \quad \text{for a.e. } s \in (0, \widehat{T})$$

En. id: $$\mathcal{E}(\widehat{t}(s_2), \widehat{u}(s_2)) - \mathcal{E}(\widehat{t}(s_1), \widehat{u}(s_1)) = \int_{s_1}^{s_2} \Big(\partial_t \mathcal{E}(\widehat{t}(s), \widehat{u}(s)) \widehat{t}'(s) - |\widehat{u}'|(s)\, |\partial \mathcal{E}|\, (\widehat{t}(s), \widehat{u}(s)) \Big)\, \mathrm{d}s \quad \forall\, 0 \leq s_1 \leq s_2 \leq \widehat{T}$$

Differential conditions: $$\begin{cases} |\widehat{u}'|(s) = 0 & \Rightarrow\ |\partial \mathcal{E}|\, (\widehat{t}(s), \widehat{u}(s)) \leq 1 \\ \widehat{t}'(s)\, |\widehat{u}'|(s) > 0 & \Rightarrow\ |\partial \mathcal{E}|\, (\widehat{t}(s), \widehat{u}(s)) = 1 \\ \widehat{t}'(s) = 0 & \Rightarrow\ |\partial \mathcal{E}|\, (\widehat{t}(s), \widehat{u}(s)) \geq 1 \end{cases}$$

Viscous slip: jump set

$$\mathcal{J} := \left\{ s_0 \in (0, \widehat{T}) \,:\, |\widehat{t}'|(s) = 0 \text{ in a neighb. of } s_0 \right\}$$

- $\widehat{t}'(s) = 0 \ \Rightarrow\ |\widehat{u}'|(s) > 0 \ \Rightarrow\ |\partial \mathcal{E}|\, (\widehat{t}(s), \widehat{u}(s)) \geq 1$
- in a neighb. $I(s_0)$ $\widehat{t}(s) \equiv \widehat{t}(s_0)$ & **energy identity** $\forall\, s_1 \leq s_2 \in I(s_0)$

$$\mathcal{E}(\widehat{t}(s_0), \widehat{u}(s_2)) - \mathcal{E}(\widehat{t}(s_0), \widehat{u}(s_1)) = -\int_{s_1}^{s_2} |\partial \mathcal{E}|\, (\widehat{t}(s_0), \widehat{u}(s))\, |\widehat{u}'|(s) \Big)\, \mathrm{d}s$$

From virtual to real jumps

With a suitable transformation,

Metric Parametrized Rate-Independent Flows

$$\textbf{En. id.:}\ \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\widehat{t}(s),\widehat{u}(s)) - \partial_t\mathcal{E}(\widehat{t}(s),\widehat{u}(s))\,\widehat{t}'(s) = -|\widehat{u}'|(s)\ |\partial\mathcal{E}|\left(\widehat{t}(s),\widehat{u}(s)\right)$$

$$\textbf{Three regimes:}\ \begin{cases} |\widehat{u}'|(s)=0 & \Rightarrow\ |\partial\mathcal{E}|\left(\widehat{t}(s),\widehat{u}(s)\right)\le 1\\ \widehat{t}'(s)\,|\widehat{u}'|(s)>0 & \Rightarrow\ |\partial\mathcal{E}|\left(\widehat{t}(s),\widehat{u}(s)\right)= 1\\ \widehat{t}'(s)=0 & \Rightarrow\ |\partial\mathcal{E}|\left(\widehat{t}(s),\widehat{u}(s)\right)\ge 1\end{cases}$$

$$\Downarrow\quad\Uparrow$$

BV (non parametrized) Rate-Independent Flows

$$\textbf{En. id.:}\ \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,u(t)) - \partial_t\mathcal{E}(t,u(t)) = -\Sigma(t,u(t-),u(t+))\cdot\mu_u \qquad \text{in } \mathcal{D}'(0,T)$$

$$\textbf{Three regimes:}\ \begin{cases} t\in[0,T]\setminus J_u \ \Rightarrow\ |\partial\mathcal{E}|\,(t,u(t))\le 1\\ t\in[0,T]\in\mathrm{supp}(\mu_u)\setminus J_u \ \Rightarrow\ |\partial\mathcal{E}|\,(t,u(t))= 1\\ t\in J_u \ \Rightarrow\ \begin{cases}\exists y\in\mathrm{AC}([0,1],X) \text{ and } \theta\in[0,1] \text{ s. t.}\\ y(0)=u(t-),\, u(\theta)=u(t),\, y(1)=u(t+),\\ |\partial\mathcal{E}|\,(t,y(r))\ge 1 \quad \text{for a.e. } r\in(0,1),\\ \mathcal{E}(t,u(t+))-\mathcal{E}(t,u(t-)) = -\int_0^1 |\partial\mathcal{E}|\,(t,y(r))|y'|(r)\mathrm{d}r.\end{cases}\end{cases}$$

Local vs. Global Slope

- **Setting:** (X, d) complete metric space

Global slope

Given $\mathcal{E} : [0, T] \times X \to (-\infty, +\infty]$ and $u \in D(\mathcal{E}(t, \cdot))$, the **global slope** of $\mathcal{E}(t, \cdot)$ at u is

$$\left|\mathcal{G}\ell(\mathcal{E})\right|(t, u) := \sup_{v \neq u} \frac{(\mathcal{E}(t, u) - \mathcal{E}(t, v))^+}{d(u, v)}$$

Suppose that $\mathcal{E}(t, \cdot)$ is λ–(geodesically) convex, $\lambda \geq 0$. Then

$$|\partial\mathcal{E}|(t, u) = \mathcal{G}\ell(\mathcal{E})(t, u)$$

Comparison with the energetic formulation

During **sliding regime** (rate-independent) we have

$$|\partial\mathcal{E}|\,(t,u(t)) = 1 \qquad \textbf{local stability}$$

For **global energetic solutions** we would have

$$|\mathcal{G}\ell(\mathcal{E})|(t,u(t)) = 1 \qquad \textbf{global stability}$$

BUT global energetic solutions jump too early!

Local vs. Global stability

A one-dimensional example

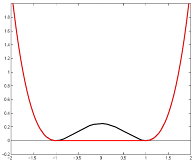
- Metric setting: $(\mathbb{R}, d_\eta)$, $d_\eta(u,v) := \eta|v-u|$, $\eta > 0$.
- Energy:

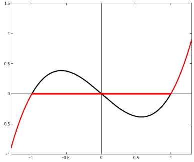
$$\mathcal{E}(t,u) = \overbrace{\frac{1}{4}(u^2-1)^2}^{\mathcal{E}(u)\ \text{double-well potential}} - \overbrace{f(t)}^{\text{(pcw.-)monotone input}}\ u$$

  to fix ideas: $f(t) := t_0 + t$, $t \in [0,T]$.
- Initial state: $u_0 \in [-\sqrt{2}, -1)$

Global Energetic evolution

Convexified energy $\mathcal{E}^{**}_{u_0}$ on $(u_0, +\infty)$ with derivative $\mathrm{D}\mathcal{E}^{**}_{u_0}$





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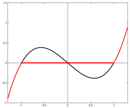
  to fix ideas: $f(t) := t_0 + t$, $t \in [0,T]$.
- Initial state: $u_0 \in [-\sqrt{2}, -1)$

Global Energetic evolution

the evolution of u is

$$u(t) = \mathrm{D}\mathcal{E}_{u_0}^{**\,-1}\left(\max\{(f(t)-\eta), \partial\mathcal{E}_{u_0}^{**}(u_0)\}\right) \qquad t > 0$$

i.e. **Maxwell rule** ($\leftrightarrow$ convexification) with a delay of η and u jumps early



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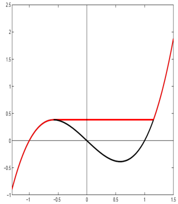
to fix ideas: $f(t) := t_0 + t$, $t \in [0,T]$.

- Initial state: $u_0 \in [-\sqrt{2}, -1)$

Metric Parametrized Rate-Independent Evolution

Minimal non decreasing graph $M(\mathcal{E}')_{u_0}$ s. t. $M(\mathcal{E}')_{u_0}(u) \geq \mathcal{E}'(u)$ for $u \geq u_0$



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- Energy:

$$\mathcal{E}(t,u) = \overbrace{\frac{1}{4}(u^2-1)^2}^{\mathcal{E}(u)\text{ double-well potential}} - \overbrace{f(t)}^{\text{(pcw.-)monotone input}} u$$

  to fix ideas: $f(t) := t_0 + t$, $t \in [0,T]$.
- Initial state: $u_0 \in [-\sqrt{2}, -1)$

Metric Parametrized Rate-Independent Evolution

the evolution of u is

$$u(t) = M(\mathcal{E}')_{u_0}{}^{-1}\left(\max\{(f(t)-\eta), \mathcal{E}'(u_0)\}\right) \qquad t > 0$$

i.e. **Hysteresis behaviour** with a delay of η and u jumps later