Analysis of doubly nonlinear evolution equations driven by nonconvex energies

Riccarda Rossi
(Università di Brescia)

in collaboration with
Alexander Mielke (WIAS & Humboldt-Universität – Berlin)
Giuseppe Savaré (Università di Pavia)

Univerzita Karlova v Praze, 11.04.2011

Doubly nonlinear evolution equations

$$\partial\Psi(z'(t)) + \partial_z\mathcal{E}(t, z(t)) \ni 0 \quad \text{in } B', \quad t \in (0, T), \tag{DNE}$$

- B is a **reflexive** Banach space;
- $\Psi : B \to [0, +\infty]$, with $\Psi(0) = 0$, l.s.c. and **convex**
- ∂ **convex analysis** subdifferential;
- $\mathcal{E} : [0, T] \times B \to (-\infty, +\infty]$, such that $z \mapsto \mathcal{E}(t, z)$ is **l.s.c.**
- ∂_z is the "subdifferential" of $\mathcal{E}$ **w.r.t. the second variable**:

Doubly nonlinear evolution equations

$$\partial\Psi(z'(t)) + \mathrm{D}_z\mathcal{E}(t, z(t)) \ni 0 \quad \text{in } B', \quad t \in (0, T), \tag{DNE}$$

- B is a **reflexive** Banach space;
- $\Psi : B \to [0, +\infty]$, with $\Psi(0) = 0$, l.s.c. and **convex**
- ∂ **convex analysis** subdifferential;
- $\mathcal{E} : [0, T] \times B \to (-\infty, +\infty]$, such that $z \mapsto \mathcal{E}(t, z)$ is **l.s.c.**
- ∂_z is the "subdifferential" of $\mathcal{E}$ **w.r.t. the second variable**:
  - **Fréchet differential** $\mathrm{D}_z\mathcal{E}$ if $z \mapsto \mathcal{E}(t, z)$ is **smooth**,

Doubly nonlinear evolution equations

$$\partial\Psi(z'(t)) + \boldsymbol{\partial_z}^{\mathbf{co}}\mathcal{E}(t, z(t)) \ni 0 \quad \text{in } B', \quad t \in (0, T), \qquad \text{(DNE)}$$

- B is a **reflexive** Banach space;
- $\Psi : B \to [0, +\infty]$, with $\Psi(0) = 0$, l.s.c. and **convex**
- ∂ **convex analysis** subdifferential;
- $\mathcal{E} : [0, T] \times B \to (-\infty, +\infty]$, such that $z \mapsto \mathcal{E}(t, z)$ is **l.s.c.**
- ∂_z is the "subdifferential" of $\mathcal{E}$ **w.r.t. the second variable**:
  - **Fréchet differential** $\mathrm{D}_z\mathcal{E}$ if $z \mapsto \mathcal{E}(t, z)$ is **smooth**,
  - **convex analysis subdifferential** $\boldsymbol{\partial_z}^{\mathbf{co}}\mathcal{E}$ if $z \mapsto \mathcal{E}(t, z)$ is **convex**, i.e.

$$\xi \in \boldsymbol{\partial_z}^{\mathbf{co}}\mathcal{E}(t, z) \quad \Leftrightarrow \mathcal{E}(t, w) - \mathcal{E}(t, z) \geq \langle \xi, w - z \rangle \quad \forall\, w \in B$$

Doubly nonlinear evolution equations

$$\partial\Psi(z'(t)) + \partial_z\mathcal{E}(t, z(t)) \ni 0 \quad \text{in } B', \quad t \in (0, T), \tag{DNE}$$

- B is a **reflexive** Banach space;
- $\Psi : B \to [0, +\infty]$, with $\Psi(0) = 0$, l.s.c. and **convex**
- ∂ **convex analysis** subdifferential;
- $\mathcal{E} : [0, T] \times B \to (-\infty, +\infty]$, such that $z \mapsto \mathcal{E}(t, z)$ is **l.s.c.**
- ∂_z is the "subdifferential" of $\mathcal{E}$ **w.r.t. the second variable**:
  - **Fréchet differential** $\mathrm{D}_z\mathcal{E}$ if $z \mapsto \mathcal{E}(t, z)$ is **smooth**,
  - **convex analysis subdifferential** $\boldsymbol{\partial_z}^{\mathbf{co}}\mathcal{E}$ if $z \mapsto \mathcal{E}(t, z)$ is **convex**, i.e.

$$\xi \in \boldsymbol{\partial_z}^{\mathbf{co}}\mathcal{E}(t, z) \quad \Leftrightarrow \mathcal{E}(t, w) - \mathcal{E}(t, z) \geq \langle \xi, w - z\rangle \quad \forall\, w \in B$$

Main focus on:

Nonsmooth, nonconvex energies

Physical interpretation

$$\partial\Psi(z'(t)) + \partial_z\mathcal{E}(t, z(t)) \ni 0 \quad \text{in } B', \quad t \in (0, T), \tag{DNE}$$

is a generalized **balance relation** in **Thermomechanics**:

- $\Psi_z \sim$ **dissipation** potential
- $\mathcal{E} \sim$ **energy** functional $(t \mapsto \mathcal{E}(t, z) \sim$ (power of) **external forces**)

Physical interpretation

$$\partial\Psi(z'(t)) + \partial_z\mathcal{E}(t, z(t)) \ni 0 \quad \text{in } B', \quad t \in (0, T), \qquad \text{(DNE)}$$

is a generalized **balance relation** in **Thermomechanics**:

- $\Psi_z \sim$ **dissipation** potential
- $\mathcal{E} \sim$ **energy** functional $(t \mapsto \mathcal{E}(t, z) \sim$ (power of) **external forces**)

Two cases:

- Ψ has **superlinear growth** $\leftrightarrow$ dissipation with **viscosity** effects

$$\lim_{\|v\| \to +\infty} \frac{\Psi(v)}{\|v\|} = +\infty$$

Physical interpretation

$$\partial\Psi(z'(t)) + \partial_z\mathcal{E}(t, z(t)) \ni 0 \quad \text{in } B', \quad t \in (0, T), \qquad \text{(DNE)}$$

is a generalized **balance relation** in **Thermomechanics**:

- $\Psi_z \sim$ **dissipation** potential
- $\mathcal{E} \sim$ **energy** functional $(t \mapsto \mathcal{E}(t, z) \sim$ (power of) **external forces**)

Two cases:

- Ψ has **superlinear growth** $\leftrightarrow$ dissipation with **viscosity** effects

$$\lim_{\|v\| \to +\infty} \frac{\Psi(v)}{\|v\|} = +\infty$$

- Ψ has **linear growth** and is **positively 1-homogeneous**

$$\Psi(\lambda v) = \lambda\Psi(v) \qquad \forall\, \lambda \geq 0\ \forall\, v \in B$$

$\leftrightarrow$ **rate-independent** models

Physical interpretation

$$\partial\Psi(z'(t)) + \partial_z\mathcal{E}(t, z(t)) \ni 0 \quad \text{in } B', \quad t \in (0, T), \qquad \text{(DNE)}$$

is a generalized **balance relation** in **Thermomechanics**:

- $\Psi_z \sim$ **dissipation** potential
- $\mathcal{E} \sim$ **energy** functional $(t \mapsto \mathcal{E}(t, z) \sim$ (power of) **external forces**)

Two cases:

- Ψ has **superlinear growth** $\leftrightarrow$ dissipation with **viscosity** effects

$$\lim_{\|v\| \to +\infty} \frac{\Psi(v)}{\|v\|} = +\infty$$

- Ψ has **linear growth** and is **positively 1-homogeneous**

$$\Psi(\lambda v) = \lambda\Psi(v) \qquad \forall\, \lambda \geq 0\ \forall\, v \in B$$

$\leftrightarrow$ **rate-independent** models

- From now ON, we take

Ψ with **superlinear growth**

The convex & autonomous case

$$\boxed{\mathcal{E}(t,z) = \mathcal{E}(z)} \quad \& \quad \mathcal{E} \textbf{ convex}$$

Then,

$$\partial\Psi(z'(t)) + \partial_z{}^{\mathbf{co}}\mathcal{E}(z(t)) \ni 0 \quad \text{in } B', \quad t \in (0,T), \qquad \text{(DNE)}$$

Applications

- phase transitions
- elasto-visco-plasticity
- gas flow through porous media....

The convex & autonomous case

$$\boxed{\mathcal{E}(t,z) = \mathcal{E}(z)} \quad \& \quad \mathcal{E} \textbf{ convex}$$

Then,

$$\partial\Psi(z'(t)) + \partial_z^{\mathbf{co}}\mathcal{E}(z(t)) \ni 0 \quad \text{in } B', \quad t \in (0,T), \qquad \text{(DNE)}$$

Applications

- phase transitions
- elasto-visco-plasticity
- gas flow through porous media....

Existence results

- existence & approximation of solutions: **[Barbu '75], [Arai '79], [Semba '86], [Colli-Visintin '90], [Colli '92]**
- existence can be extended to C^1- (or lower order) **perturbations** of convex energies **[Segatti '06], [Schimperna-Segatti-Stefanelli '07]**

Reduced energy functionals

- ♣ Consider **marginal energy functionals**

$$\mathcal{E}(t,z) := \min_{\eta \in \mathcal{F}} I(t,\eta,z)$$

Reduced energy functionals

♣ Consider **marginal energy functionals**

$$\mathcal{E}(t,z) := \min_{\eta\in\mathcal{F}} I(t,\eta,z)$$

To fix ideas, suppose that

$$\begin{aligned} \eta \mapsto I(t,\eta,z) &\quad \text{is smooth} \\ z \mapsto I(t,\eta,z) &\quad \text{is smooth} \end{aligned}$$

Reduced energy functionals

♣ Consider **marginal energy functionals**

$$\mathcal{E}(t,z) := \min_{\eta \in \mathcal{F}} I(t,\eta,z)$$

To fix ideas, suppose that

$$\begin{aligned} \eta &\mapsto I(t,\eta,z) \quad &&\text{is smooth} \\ z &\mapsto I(t,\eta,z) \quad &&\text{is smooth} \end{aligned}$$

♢ For a **suitable** subdifferential notion $\partial_z \mathcal{E}$,

$$\partial\Psi(z'(t)) + \partial_z \mathcal{E}(t,z(t)) \ni 0 \quad \text{in } B', \quad t \in (0,T), \qquad \text{(DNE)}$$

yields solutions to:

The associated PDE system

$$\begin{cases} \partial\Psi(z'(t)) + \mathrm{D}_z I(t,\eta(t),z(t)) = 0, \\ \mathrm{D}_\eta I(t,\eta(t),z(t)) = 0 \end{cases} \quad \text{in } B^*, \quad \text{for a.a. } t \in (0,T),$$

where

- z is the **internal variable**, through which **dissipation** occurs
- η fulfils **stationary** equation

Nonconvexity & nonsmoothness

$$\mathcal{E}(t,z) := \min_{\eta \in \mathcal{F}} I(t,\eta,z)$$

- $z \mapsto \mathcal{E}(t,z)$ is (highly) **nonconvex**

Which notion for the subdifferential $\partial_z \mathcal{E}$???

Nonconvexity & nonsmoothness

$$\mathcal{E}(t,z) := \min_{\eta \in \mathcal{F}} I(t,\eta,z)$$

- $z \mapsto \mathcal{E}(t,z)$ is (highly) **nonconvex**

  Which notion for the subdifferential $\partial_z \mathcal{E}$???

- $t \mapsto \mathcal{E}(t,z)$ is **nonsmooth**

  Which notion for $\partial_t \mathcal{E}$???

Nonconvexity & nonsmoothness

$$\mathcal{E}(t,z) := \min_{\eta \in \mathcal{F}} I(t,\eta,z)$$

- $z \mapsto \mathcal{E}(t,z)$ is (highly) **nonconvex**

Which notion for the subdifferential $\partial_z \mathcal{E}$???

- $t \mapsto \mathcal{E}(t,z)$ is **nonsmooth**

Which notion for $\partial_t \mathcal{E}$???

Main goal

$$\partial\Psi(z'(t)) + \partial_z \mathcal{E}(t, z(t)) \ni 0 \quad \text{in } B', \quad t \in (0,T), \qquad \text{(DNE)}$$

- deal with energies s.t. $z \mapsto \mathcal{E}(t,z)$ is **nonsmooth**, **nonconvex** & $t \mapsto \mathcal{E}(t,z)$ is **nonsmooth**
- use suitable **generalized** notions for $\partial_z \mathcal{E}$ and $\partial_t \mathcal{E}$
- find **sufficient conditions on $(\Psi, \mathcal{E}, \partial_z \mathcal{E})$ for existence** of solutions to (DNE)

Overview

$$\partial\Psi(z'(t)) + \partial_z\mathcal{E}(t, z(t)) \ni 0 \quad \text{in } B', \quad t \in (0, T), \qquad \text{(DNE)}$$

Overview

$$z'(t) + \partial_z \mathcal{E}(z(t)) \ni 0 \quad \text{in } \mathcal{H}, \quad t \in (0,T), \tag{GF}$$

Outline

1. consider

gradient flow case $\Psi(v) = \frac{1}{2}\|v\|^2$, $B = \mathcal{H}$ Hilbert space
automous case $\mathcal{E}(t,z) = \mathcal{E}(z)$

but with $z \mapsto \mathcal{E}(z)$ **nonconvex, nonsmooth**

Overview

$$\partial\Psi(z'(t)) + \partial_z\mathcal{E}(t, z(t)) \ni 0 \quad \text{in } B', \quad t \in (0, T), \qquad \text{(DNE)}$$

Outline

1. consider

   gradient flow case $\Psi(v) = \frac{1}{2}\|v\|^2$, $B = \mathcal{H}$ Hilbert space
   automous case $\mathcal{E}(t, z) = \mathcal{E}(z)$

   but with $z \mapsto \mathcal{E}(z)$ **nonconvex, nonsmooth**

2. consider

   doubly nonlinear case Ψ general, superlinear growth
   nonautomous case $\mathcal{E} = \mathcal{E}(t, z)$
   nonconvex, nonsmooth $z \mapsto \mathcal{E}(t, z)$

   but with $t \mapsto \mathcal{E}(t, z)$ **smooth**

Overview

$$\partial\Psi(z'(t)) + \partial_z\mathcal{E}(t, z(t)) \ni 0 \quad \text{in } B', \quad t \in (0, T), \qquad \text{(DNE)}$$

Outline

1. consider

gradient flow case	$\Psi(v) = \frac{1}{2}\|v\|^2$, $B = \mathcal{H}$ Hilbert space
automous case	$\mathcal{E}(t, z) = \mathcal{E}(z)$

   but with $z \mapsto \mathcal{E}(z)$ **nonconvex, nonsmooth**

2. consider

doubly nonlinear case	Ψ general, superlinear growth
nonautomous case	$\mathcal{E} = \mathcal{E}(t, z)$
nonconvex, nonsmooth	$z \mapsto \mathcal{E}(t, z)$

   but with $t \mapsto \mathcal{E}(t, z)$ **smooth**

3. **general** case with $t \mapsto \mathcal{E}(t, z)$ **nonsmooth**: e.g., **marginal energies**

Overview

$$\partial\Psi(z'(t)) + \partial_z\mathcal{E}(t,z(t)) \ni 0 \quad \text{in } B', \quad t \in (0,T), \qquad \text{(DNE)}$$

Outline

1. consider

gradient flow case	$\Psi(v) = \frac{1}{2}\|v\|^2$, $B = \mathcal{H}$ Hilbert space
automous case	$\mathcal{E}(t,z) = \mathcal{E}(z)$

   but with $z \mapsto \mathcal{E}(z)$ **nonconvex, nonsmooth**

2. consider

doubly nonlinear case	Ψ general, superlinear growth
nonautomous case	$\mathcal{E} = \mathcal{E}(t,z)$
nonconvex, nonsmooth	$z \mapsto \mathcal{E}(t,z)$

   but with $t \mapsto \mathcal{E}(t,z)$ **smooth**

3. **general** case with $t \mapsto \mathcal{E}(t,z)$ **nonsmooth**: e.g., **marginal energies**
4. **application**

Gradient flows of convex functionals: results

♣ Suppose that $\mathcal{E}(t,z) = \mathcal{E}(z)$, with $\mathcal{E}$ **convex**:

$$\begin{cases} z'(t) + \partial_z{}^{\mathbf{co}}\mathcal{E}(z(t)) \ni 0 \quad \text{in } \mathcal{H}, \quad t \in (0,T) \\ z(0) = z_0, \end{cases}$$

- **Well-established literature on:** existence, uniqueness, approximation of solutions **[Kōmura'67, Crandall-Pazy'69, Brézis'73..]**

Gradient flows of convex functionals: results

♣ Suppose that $\mathcal{E}(t,z) = \mathcal{E}(z)$, with $\mathcal{E}$ **convex**:

$$\begin{cases} z'(t) + \partial_z{}^{\mathbf{co}}\mathcal{E}(z(t)) \ni 0 \ \text{ in } \mathcal{H}, \quad t \in (0,T) \\ z(0) = z_0, \end{cases}$$

- **Well-established literature on:** existence, uniqueness, approximation of solutions **[Kōmura'67, Crandall-Pazy'69, Brézis'73..]**
- How extend (some of) these results to the **nonconvex** case?
- **Idea:** Let's have a deeper look at the **convex** case!

Gradient flows of convex functionals: approximation

- Fix time step $\tau > 0 \quad \rightsquigarrow \quad$ partition of $(0, T)$
- **Discrete solutions** $Z_\tau^0, Z_\tau^1, \ldots, Z_\tau^N$: solve recursively

$$Z_\tau^n \in \mathrm{Argmin}_{z \in \mathcal{H}} \left\{ \frac{1}{2\tau} \| z - Z_\tau^{n-1} \|^2 + \mathcal{E}(z) \right\}, \quad Z_\tau^0 := z_0$$

Gradient flows of convex functionals: approximation

- Fix time step $\tau > 0 \rightsquigarrow$ partition of $(0, T)$
- **Discrete solutions** $Z_\tau^0, Z_\tau^1, \ldots, Z_\tau^N$: solve recursively

$$\textbf{Euler equation:} \quad \frac{Z_\tau^n - Z_\tau^{n-1}}{\tau} + \boldsymbol{\partial}_z{}^{\mathbf{co}}\mathcal{E}(Z_\tau^n) \ni 0, \quad Z_\tau^0 := z_0$$

- **Approximate solutions**: interpolants on $(0, T)$ of $\{Z_\tau^k\}_{k=1}^n$:

$$\{\overline{Z}_\tau\} \text{ piecewise constant;}$$
$$\{\widehat{Z}_\tau\} \text{ piecewise linear.}$$

- **Approximate equation**:

$$\widehat{Z}_\tau'(t) + \boldsymbol{\partial}_z{}^{\mathbf{co}}\mathcal{E}(\overline{Z}_\tau(t)) \ni 0 \quad t \in (0, T)$$

Gradient flows of convex functionals: approximation

- Fix time step $\tau > 0 \quad \rightsquigarrow \quad$ partition of $(0, T)$
- **Discrete solutions** $Z_\tau^0, Z_\tau^1, \ldots, Z_\tau^N$: solve recursively

$$\textbf{Euler equation:} \quad \frac{Z_\tau^n - Z_\tau^{n-1}}{\tau} + \partial_z{}^{\mathbf{co}} \mathcal{E}(Z_\tau^n) \ni 0, \quad Z_\tau^0 := z_0$$

- **Approximate solutions**: interpolants on $(0, T)$ of $\{Z_\tau^k\}_{k=1}^n$:

$\{\overline{Z}_\tau\}$ piecewise constant;
$\{\widehat{Z}_\tau\}$ piecewise linear.

- **Approximate equation**:

$$\widehat{Z}_\tau'(t) + \partial_z{}^{\mathbf{co}} \mathcal{E}(\overline{Z}_\tau(t)) \ni 0 \quad t \in (0, T)$$

- **Approximate energy inequality**:

$$\int_0^t \|\widehat{Z}_\tau'(s)\|^2 \, \mathrm{d}s + \mathcal{E}(\overline{Z}_\tau(t)) \leq \mathcal{E}(z_0).$$

$\Rightarrow$ **A priori estimates** for $\{\overline{Z}_\tau\}$, $\{\widehat{Z}_\tau\}$

Gradient flows of convex functionals: existence

- **Compactness**: $\exists\, z \in H^1(0,T;\mathcal{H})$

$$\begin{cases} \overline{Z}_\tau,\, \widehat{Z}_\tau \to z & \text{in } L^\infty(0,T;\mathcal{H}), \\ \widehat{Z}_\tau' \rightharpoonup z' & \text{in } H^1(0,T;\mathcal{H}) \end{cases}$$

Gradient flows of convex functionals: existence

- **Compactness**: $\exists\, z \in H^1(0,T;\mathcal{H})$

$$\begin{cases} \overline{Z}_\tau, \widehat{Z}_\tau \to z & \text{in } L^\infty(0,T;\mathcal{H}), \\ \widehat{Z}_\tau' \rightharpoonup z' & \text{in } H^1(0,T;\mathcal{H}) \end{cases}$$

- **Passage to the limit based on :**

convexity: the set $\boldsymbol{\partial_z}^{\mathbf{co}}\mathcal{E}(z)$ is **convex** and closed

closedness: graph of $\boldsymbol{\partial_z}^{\mathbf{co}}\mathcal{E}$ is seq. closed w.r.t. **strong-weak** top. in $\mathcal{H} \times \mathcal{H}$:

$$z_n \to z, \quad \xi_n \rightharpoonup \xi, \quad \xi_n \in \boldsymbol{\partial_z}^{\mathbf{co}}\mathcal{E}(z_n)\ \forall n \in \mathbb{N} \quad \Rightarrow \quad \xi \in \boldsymbol{\partial_z}^{\mathbf{co}}\mathcal{E}(z)$$

- **Conclusion:**

$$-\widehat{Z}_\tau'(t) \in \boldsymbol{\partial_z}^{\mathbf{co}}\mathcal{E}(\overline{Z}_\tau(t)), \quad t \in (0,T)$$

$$\lim_{\tau \downarrow 0}$$

$$-z'(t) \in \boldsymbol{\partial_z}^{\mathbf{co}}\mathcal{E}(z(t)), \quad t \in (0,T)$$

Gradient flows of convex functionals: energy identity

Since $\partial_z{}^{\mathbf{co}}\mathcal{E}$ fulfils the **chain rule**

$$\begin{cases} z \in H^1(0,T;\mathcal{H}), \\ \xi \in L^2(0,T;\mathcal{H}), \\ \xi(t) \in \partial_z{}^{\mathbf{co}}\mathcal{E}(z(t)) \text{ a.e. in } (0,T) \end{cases} \Rightarrow \frac{d}{dt}\mathcal{E}(z(t)) = \langle \xi(t), z'(t) \rangle \text{ a.e. in } (0,T)$$

$\Rightarrow$ the solution z satisfies the **energy identity**:

$$\int_s^t \|z'(r)\|^2 \, \mathrm{d}r + \mathcal{E}(z(t)) = \mathcal{E}(z(s)) \quad \text{for all } 0 \leq s \leq t \leq T.$$

From convex to non convex: heuristics

♠ Now, $\mathcal{E} = \mathcal{E}(z)$ is **nonconvex**

From convex to non convex: heuristics

♠ Now, $\mathcal{E} = \mathcal{E}(z)$ is **nonconvex**

Tentative approximation

- Fix time step $\tau > 0 \rightsquigarrow$ partition of $(0, T)$
- **Discrete solutions:** the recursive minimization problem

$$\text{Find } Z_\tau^0 := z_0, \qquad Z_\tau^1, \ldots, Z_\tau^N:$$
$$Z_\tau^n \in \mathrm{Argmin}_{z \in \mathcal{H}} \left\{ \frac{1}{2\tau} \|z - Z_\tau^{n-1}\|^2 + \mathcal{E}(z) \right\}$$

has a solution if $\mathcal{E}$ is **coercive** (for instance, $\mathcal{E}(\cdot) + \|\cdot\|^2$ has **compact** sublevels)

- Discrete solutions fulfil the **Euler equation**

$$\frac{Z_\tau^n - Z_\tau^{n-1}}{\tau} + \partial_z{}^{\mathrm{fr}} \mathcal{E}(Z_\tau^n) \ni 0,$$

with $\partial_z{}^{\mathrm{fr}} \mathcal{E}$ the **Fréchet subdifferential** of $\mathcal{E}$

The Fréchet subdifferential

First idea: "localize" the convex subdifferential

The convex subdifferential

Given $z \in D(\mathcal{E})$,

$$\xi \in \partial_z{}^{\mathbf{co}}\mathcal{E}(z) \quad \Leftrightarrow \mathcal{E}(w) - \mathcal{E}(z) \geq \langle \xi, w - z \rangle \quad \text{for all } w \in \mathcal{H}$$

The Fréchet subdifferential

First idea: "localize" the convex subdifferential

The convex subdifferential

Given $z \in D(\mathcal{E})$,

$$\xi \in \partial_z{}^{\mathbf{co}}\mathcal{E}(z) \quad \Leftrightarrow \mathcal{E}(w) - \mathcal{E}(z) \geq \langle \xi, w - z \rangle \quad \textbf{for all } w \in \mathcal{H}$$

The Fréchet subdifferential

Given $z \in D(\mathcal{E})$,

$$\xi \in \partial_z{}^{\mathbf{fr}}\mathcal{E}(z) \quad \Leftrightarrow \liminf_{w \to z} \frac{\mathcal{E}(w) - \mathcal{E}(z) - \langle \xi, w - z \rangle}{\|w - z\|} \geq 0$$

The Fréchet subdifferential

First idea: "localize" the convex subdifferential

The convex subdifferential

Given $z \in D(\mathcal{E})$,

$$\xi \in \partial_z{}^{\mathbf{co}}\mathcal{E}(z) \quad \Leftrightarrow \mathcal{E}(w) - \mathcal{E}(z) \geq \langle \xi, w - z \rangle \quad \textbf{for all } w \in \mathcal{H}$$

The Fréchet subdifferential

Given $z \in D(\mathcal{E})$,

$$\xi \in \partial_z{}^{\mathbf{fr}}\mathcal{E}(z) \quad \Leftrightarrow \liminf_{w \to z} \frac{\mathcal{E}(w) - \mathcal{E}(z) - \langle \xi, w - z \rangle}{\|w - z\|} \geq 0$$

♣ If $\mathcal{E}$ is **convex**, then $\partial_z{}^{\mathbf{fr}}\mathcal{E} \equiv \partial_z{}^{\mathbf{co}}\mathcal{E}$

From convex to non convex: heuristics

- **Discrete solutions**: $Z_\tau^0 = z_0, \qquad Z_\tau^1, \ldots, Z_\tau^N \Rightarrow$ approximate solutions

From convex to non convex: heuristics

- **Discrete solutions**: $Z_\tau^0 = z_0, \qquad Z_\tau^1, \ldots, Z_\tau^N \Rightarrow$ approximate solutions
- **Approximate energy inequality**: in the nonconvex case it is not clear how to prove

$$\int_0^t \|\widehat{Z}_\tau'(s)\|^2 \,\mathrm{d}s + \mathcal{E}(\overline{Z}_\tau(t)) \leq \mathcal{E}(z_0).$$

To obtain *some* energy inequality, need to work with a **suitable interpolant** of $Z_\tau^0 = z_0$, Z_τ^1, ..., Z_τ^N.

From convex to non convex: heuristics

- **Discrete solutions**: $Z_\tau^0 = z_0, \qquad Z_\tau^1, \ldots, Z_\tau^N \Rightarrow$ approximate solutions
- **Approximate energy inequality**: in the nonconvex case it is not clear how to prove

$$\int_0^t \|\widehat{Z}_\tau'(s)\|^2 \, \mathrm{d}s + \mathcal{E}(\overline{Z}_\tau(t)) \leq \mathcal{E}(z_0).$$

To obtain *some* energy inequality, need to work with a **suitable interpolant** of $Z_\tau^0 = z_0, \, Z_\tau^1, \, \ldots, \, Z_\tau^N$.

The variational interpolant [E. DeGiorgi, theory of Minimzing Movements, Gradient Flows in Metric Spaces]

Defined by

$$\widetilde{Z}_\tau(0) = Z_\tau^0 = z_0,$$

$$\widetilde{Z}_\tau(t) \in \mathrm{Argmin}_{z \in \mathcal{H}} \left\{ \frac{1}{2(t - t_{n-1})} \|z - Z_\tau^{n-1}\|^2 + \mathcal{E}(z) \right\} \text{ for } t \in (t_{n-1}, t_n]$$

From convex to non convex: heuristics

- **Discrete solutions**: $Z_\tau^0 = z_0, \qquad Z_\tau^1, \ldots, Z_\tau^N \Rightarrow$ approximate solutions
- **Approximate energy inequality**: in the nonconvex case it is not clear how to prove

$$\int_0^t \|\widehat{Z}_\tau'(s)\|^2 \,\mathrm{d}s + \mathcal{E}(\overline{Z}_\tau(t)) \leq \mathcal{E}(z_0).$$

To obtain *some* energy inequality, need to work with a **suitable interpolant** of $Z_\tau^0 = z_0$, Z_τ^1, $\ldots$, Z_τ^N.

The variational interpolant [E. DeGiorgi, theory of Minimzing Movements, Gradient Flows in Metric Spaces]

Defined by

$$\widetilde{Z}_\tau(0) = Z_\tau^0 = z_0,$$

$$\widetilde{Z}_\tau(t) \in \mathrm{Argmin}_{z \in \mathcal{H}} \left\{ \frac{1}{2(t - t_{n-1})} \|z - Z_\tau^{n-1}\|^2 + \mathcal{E}(z) \right\} \text{ for } t \in (t_{n-1}, t_n]$$

Compare with the **time-incremental minimization**

$$\mathrm{Argmin}_{z \in \mathcal{H}} \left\{ \frac{1}{2\tau} \|z - Z_\tau^{n-1}\|^2 + \mathcal{E}(z) \right\}$$

From convex to non convex: heuristics

- **Discrete solutions**: $Z_\tau^0 = z_0, \qquad Z_\tau^1, \ldots, Z_\tau^N \Rightarrow$ approximate solutions
- **Approximate energy inequality**: in the nonconvex case it is not clear how to prove

$$\int_0^t \|\widehat{Z}_\tau'(s)\|^2 \,\mathrm{d}s + \mathcal{E}(\overline{Z}_\tau(t)) \leq \mathcal{E}(z_0).$$

To obtain *some* energy inequality, need to work with a **suitable interpolant** of $Z_\tau^0 = z_0$, Z_τ^1, ..., Z_τ^N.

The variational interpolant [E. DeGiorgi, theory of Minimzing Movements, Gradient Flows in Metric Spaces]

Defined by

$$\widetilde{Z}_\tau(0) = Z_\tau^0 = z_0,$$

$$\widetilde{Z}_\tau(t) \in \mathrm{Argmin}_{z\in\mathcal{H}} \left\{ \frac{1}{2(t-t_{n-1})} \|z - Z_\tau^{n-1}\|^2 + \mathcal{E}(z) \right\} \text{ for } t \in (t_{n-1}, t_n]$$

Euler equation

$$\frac{\widetilde{Z}_\tau(t) - Z_\tau^{n-1}}{(t-t_{n-1})} + \partial_z{}^{\mathrm{fr}} \mathcal{E}(\widetilde{Z}_\tau(t)) \ni 0, \quad t \in (0,T),$$

From convex to non convex: heuristics

- **Discrete solutions**: $Z_\tau^0 = z_0, \qquad Z_\tau^1, \ldots, Z_\tau^N \Rightarrow$ approximate solutions
- **Approximate energy inequality**: in the nonconvex case it is not clear how to prove

$$\int_0^t \|\widehat{Z}_\tau'(s)\|^2 \, ds + \mathcal{E}(\overline{Z}_\tau(t)) \leq \mathcal{E}(z_0).$$

To obtain *some* energy inequality, need to work with a **suitable interpolant** of $Z_\tau^0 = z_0, Z_\tau^1, \ldots, Z_\tau^N$.

The variational interpolant [E. DeGiorgi, theory of Minimzing Movements, Gradient Flows in Metric Spaces]

Defined by

$$\widetilde{Z}_\tau(0) = Z_\tau^0 = z_0,$$

$$\widetilde{Z}_\tau(t) \in \mathrm{Argmin}_{z \in \mathcal{H}} \left\{ \frac{1}{2(t - t_{n-1})} \|z - Z_\tau^{n-1}\|^2 + \mathcal{E}(z) \right\} \text{ for } t \in (t_{n-1}, t_n]$$

Euler equation

$$\exists\, \widetilde{\xi}_\tau(t) \in \partial_z{}^{\mathrm{fr}} \mathcal{E}(\widetilde{Z}_\tau(t)) \qquad \text{s.t.} \qquad \frac{\widetilde{Z}_\tau(t) - Z_\tau^{n-1}}{(t - t_{n-1})} + \widetilde{\xi}_\tau(t) = 0, \quad t \in (0, T).$$

First step: approximate energy inequality

Approximate energy inequality in the **convex case**:

$$\int_0^t \|\widehat{Z}_\tau'(s)\|^2 \, ds + \mathcal{E}(\overline{Z}_\tau(t)) \leq \mathcal{E}(z_0).$$

In the **nonconvex case**:

$$\frac{1}{2}\int_0^t \|\widehat{Z}_\tau'(s)\|^2 \, ds + \frac{1}{2}\int_0^t \|\widetilde{\xi}_\tau(s)\|^2 ds + \mathcal{E}(\widetilde{Z}_\tau(t)) \leq \mathcal{E}(z_0)$$

$\Rightarrow$ **A priori estimates** for $(\overline{Z}_\tau)_\tau$, $(\widehat{Z}_\tau)_\tau$, $(\widetilde{Z}_\tau)_\tau$, $(\widetilde{\xi}_\tau)_\tau$, and **compactness**:

$$\begin{aligned} &\overline{Z}_\tau,\ \widehat{Z}_\tau, \widetilde{Z}_\tau \to z && \text{in } L^\infty(0,T;\mathcal{H}),\\ &\widehat{Z}_\tau' \rightharpoonup z' && \text{in } L^2(0,T;\mathcal{H}),\\ &\widetilde{\xi}_\tau \rightharpoonup \xi && \text{in } L^2(0,T;\mathcal{H}). \end{aligned}$$

Second step: upper energy estimate

Second idea:

Instead of passing to the limit in the **pointwise** equation

$$-\widehat{Z}_\tau'(t) \in \boldsymbol{\partial_z}^{\mathbf{fr}} \mathcal{E}(\overline{Z}_\tau(t)) \quad t \in (0, T)$$

Second step: upper energy estimate

Second idea:

Instead of passing to the limit in the **pointwise** equation

$$-\widehat{Z}_\tau'(t) \in \partial_z{}^{\mathrm{fr}}\mathcal{E}(\overline{Z}_\tau(t)) \quad t \in (0, T)$$

pass to the limit in the **approximate energy inequality**

$$\frac{1}{2}\int_0^t \|\widehat{Z}_\tau'(s)\|^2 \,\mathrm{d}s + \frac{1}{2}\int_0^t \|\widetilde{\xi}_\tau(s)\|^2 \mathrm{d}s + \mathcal{E}(\widetilde{Z}_\tau(t)) \leq \mathcal{E}(z_0)$$

Second step: upper energy estimate

Second idea:

Instead of passing to the limit in the **pointwise** equation

$$-\widehat{Z}_\tau'(t) \in \boldsymbol{\partial_z}^{\mathbf{fr}}\mathcal{E}(\overline{Z}_\tau(t)) \quad t \in (0,T)$$

pass to the limit in the **approximate energy inequality**

$$\liminf_{\tau\downarrow 0}\frac{1}{2}\int_0^t \|\widehat{Z}_\tau'(s)\|^2\,\mathrm{d}s + \liminf_{\tau\downarrow 0}\frac{1}{2}\int_0^t \|\widetilde{\xi}_\tau(s)\|^2\mathrm{d}s + \liminf_{\tau\downarrow 0}\mathcal{E}(\widetilde{Z}_\tau(t)) \leq \mathcal{E}(z_0)$$

$\downarrow$ (via **LOWER SEMICONTINUITY**)

$$\frac{1}{2}\int_0^t \|z'(s)\|^2\,\mathrm{d}s + \frac{1}{2}\int_0^t \|\xi(s)\|^2\mathrm{d}s + \mathcal{E}(z(t)) \leq \mathcal{E}(z_0)$$

Second step: upper energy estimate

Second idea:
Instead of passing to the limit in the **pointwise** equation

$$-\widehat{Z}_\tau'(t) \in \boldsymbol{\partial_z}^{\mathbf{fr}} \mathcal{E}(\overline{Z}_\tau(t)) \quad t \in (0, T)$$

pass to the limit in the **approximate energy inequality**

$$\liminf_{\tau \downarrow 0} \frac{1}{2} \int_0^t \|\widehat{Z}_\tau'(s)\|^2 \, \mathrm{d}s + \liminf_{\tau \downarrow 0} \frac{1}{2} \int_0^t \|\widetilde{\xi}_\tau(s)\|^2 \mathrm{d}s + \liminf_{\tau \downarrow 0} \mathcal{E}(\widetilde{Z}_\tau(t)) \leq \mathcal{E}(z_0)$$

$\downarrow$ (via **LOWER SEMICONTINUITY**)

$$\frac{1}{2} \int_0^t \|z'(s)\|^2 \, \mathrm{d}s + \frac{1}{2} \int_0^t \|\xi(s)\|^2 \mathrm{d}s + \mathcal{E}(z(t)) \leq \mathcal{E}(z_0)$$

BUT: ¿¿ $\xi(s) \in \boldsymbol{\partial_z}^{\mathbf{fr}} \mathcal{E}(z(s))$??

Second step: upper energy estimate

Second idea:

Instead of passing to the limit in the **pointwise** equation

$$-\widehat{Z}_\tau'(t) \in \partial_z{}^{\mathbf{fr}}\mathcal{E}(\overline{Z}_\tau(t)) \quad t \in (0,T)$$

pass to the limit in the **approximate energy inequality**

$$\liminf_{\tau\downarrow 0} \frac{1}{2}\int_0^t \|\widehat{Z}_\tau'(s)\|^2\,\mathrm{d}s + \liminf_{\tau\downarrow 0} \frac{1}{2}\int_0^t \|\widetilde{\xi}_\tau(s)\|^2\mathrm{d}s + \liminf_{\tau\downarrow 0} \mathcal{E}(\widetilde{Z}_\tau(t)) \le \mathcal{E}(z_0)$$

$\downarrow$ (via **LOWER SEMICONTINUITY**)

$$\frac{1}{2}\int_0^t \|z'(s)\|^2\,\mathrm{d}s + \frac{1}{2}\int_0^t \|\xi(s)\|^2\mathrm{d}s + \mathcal{E}(z(t)) \le \mathcal{E}(z_0)$$

BUT: ¿¿ $\xi(s) \in \partial_z{}^{\mathbf{fr}}\mathcal{E}(z(s))$??

♣ **YES, if $\partial_z{}^{\mathbf{fr}}\mathcal{E}$ IS strongly-weakly closed** in the sense of graphs, i.e.

$$z_n \to z, \quad \xi_n \rightharpoonup \xi, \quad \sup_n \mathcal{E}(z_n) < +\infty, \quad \xi_n \in \partial_z{}^{\mathbf{fr}}\mathcal{E}(z_n)\ \forall n \in \mathbb{N} \quad \Rightarrow \quad \xi \in \partial_z{}^{\mathbf{fr}}\mathcal{E}(z)$$

Third step: LOWER energy estimate and conclusion

UPPER energy estimate

If $\partial_z{}^{\mathbf{fr}}\mathcal{E}$ **strongly-weakly closed**, then we have **energy inequality**

$$\frac{1}{2}\int_0^t \|z'(s)\|^2\,\mathrm{d}s + \frac{1}{2}\int_0^t \|\xi(s)\|^2\mathrm{d}s + \mathcal{E}(z(t)) \leq \mathcal{E}(z_0), \quad \xi(s) \in \partial_z{}^{\mathbf{fr}}\mathcal{E}(z(s))$$

Third step: LOWER energy estimate and conclusion

UPPER energy estimate

If ${\partial_z}^{\mathrm{fr}}\mathcal{E}$ **strongly-weakly closed**, then we have **energy inequality**

$$\frac{1}{2}\int_0^t \|z'(s)\|^2\,\mathrm{d}s + \frac{1}{2}\int_0^t \|\xi(s)\|^2\mathrm{d}s + \mathcal{E}(z(t)) \le \mathcal{E}(z_0), \quad \xi(s) \in {\partial_z}^{\mathrm{fr}}\mathcal{E}(z(s))$$

♣ **IF** ${\partial_z}^{\mathrm{fr}}\mathcal{E}$ fulfils the **chain rule**

$$\begin{cases} z \in H^1(0,T;\mathcal{H}), \\ \xi \in L^2(0,T;\mathcal{H}), \\ \xi(t) \in {\partial_z}^{\mathrm{fr}}\mathcal{E}(z(t)) \text{ a.e. in } (0,T) \end{cases} \Rightarrow \frac{d}{dt}\mathcal{E}(z(t)) = \langle \xi(t), z'(t) \rangle \text{ a.e. in } (0,T)$$

Third step: LOWER energy estimate and conclusion

UPPER energy estimate

If $\partial_z{}^{\mathrm{fr}}\mathcal{E}$ **strongly-weakly closed**, then we have **energy inequality**

$$\frac{1}{2}\int_0^t \|z'(s)\|^2\,\mathrm{d}s + \frac{1}{2}\int_0^t \|\xi(s)\|^2\mathrm{d}s + \mathcal{E}(z(t)) \leq \mathcal{E}(z_0), \quad \xi(s) \in \partial_z{}^{\mathrm{fr}}\mathcal{E}(z(s))$$

♣ **IF** $\partial_z{}^{\mathrm{fr}}\mathcal{E}$ fulfils the **chain rule**

$$\begin{cases} z \in H^1(0,T;\mathcal{H}), \\ \xi \in L^2(0,T;\mathcal{H}), \\ \xi(t) \in \partial_z{}^{\mathrm{fr}}\mathcal{E}(z(t)) \text{ a.e. in } (0,T) \end{cases} \Rightarrow \frac{d}{dt}\mathcal{E}(z(t)) = \langle \xi(t), z'(t)\rangle \text{ a.e. in } (0,T)$$

then

$$\frac{1}{2}\int_0^t \|z'(s)\|^2\,\mathrm{d}s + \frac{1}{2}\int_0^t \|\xi(s)\|^2\mathrm{d}s + \mathcal{E}(z(t)) \leq \mathcal{E}(z_0)$$

$$= \mathcal{E}(z(t)) - \int_0^t \langle \xi(s), z'(s)\rangle\,\mathrm{d}s$$

Third step: LOWER energy estimate and conclusion

UPPER energy estimate

If $\partial_z{}^{\mathrm{fr}}\mathcal{E}$ **strongly-weakly closed**, then we have **energy inequality**

$$\frac{1}{2}\int_0^t \|z'(s)\|^2\,\mathrm{d}s + \frac{1}{2}\int_0^t \|\xi(s)\|^2\mathrm{d}s + \mathcal{E}(z(t)) \leq \mathcal{E}(z_0), \quad \xi(s) \in \partial_z{}^{\mathrm{fr}}\mathcal{E}(z(s))$$

♣ **IF** $\partial_z{}^{\mathrm{fr}}\mathcal{E}$ fulfils the **chain rule**

$$\begin{cases} z \in H^1(0,T;\mathcal{H}), \\ \xi \in L^2(0,T;\mathcal{H}), \\ \xi(t) \in \partial_z{}^{\mathrm{fr}}\mathcal{E}(z(t)) \text{ a.e. in } (0,T) \end{cases} \Rightarrow \frac{d}{dt}\mathcal{E}(z(t)) = \langle \xi(t), z'(t)\rangle \text{ a.e. in } (0,T)$$

then

$$\frac{1}{2}\int_0^t \|z'(s)\|^2\,\mathrm{d}s + \frac{1}{2}\int_0^t \|\xi(s)\|^2\mathrm{d}s + \mathcal{E}(z(t)) \leq \mathcal{E}(z_0)$$

$$= \mathcal{E}(z(t)) - \int_0^t \langle \xi(s), z'(s)\rangle\,\mathrm{d}s$$

$$\Longrightarrow \quad \int_0^t \left(\|z'(s)\|^2 + \|\xi(s)\|^2 + 2\langle \xi(s), z'(s)\rangle\right)\mathrm{d}s = 0$$

Third step: LOWER energy estimate and conclusion

UPPER energy estimate

If $\partial_z{}^{\mathbf{fr}}\mathcal{E}$ **strongly-weakly closed**, then we have **energy inequality**

$$\frac{1}{2}\int_0^t \|z'(s)\|^2 \,\mathrm{d}s + \frac{1}{2}\int_0^t \|\xi(s)\|^2 \mathrm{d}s + \mathcal{E}(z(t)) \leq \mathcal{E}(z_0), \quad \xi(s) \in \partial_z{}^{\mathbf{fr}}\mathcal{E}(z(s))$$

♣ **IF** $\partial_z{}^{\mathbf{fr}}\mathcal{E}$ fulfils the **chain rule**

$$\begin{cases} z \in H^1(0,T;\mathcal{H}), \\ \xi \in L^2(0,T;\mathcal{H}), \\ \xi(t) \in \partial_z{}^{\mathbf{fr}}\mathcal{E}(z(t)) \text{ a.e. in } (0,T) \end{cases} \Rightarrow \frac{d}{dt}\mathcal{E}(z(t)) = \langle \xi(t), z'(t) \rangle \text{ a.e. in } (0,T)$$

then

$$\frac{1}{2}\int_0^t \|z'(s)\|^2 \,\mathrm{d}s + \frac{1}{2}\int_0^t \|\xi(s)\|^2 \mathrm{d}s + \mathcal{E}(z(t)) \leq \mathcal{E}(z_0)$$

$$= \mathcal{E}(z(t)) - \int_0^t \langle \xi(s), z'(s) \rangle \,\mathrm{d}s$$

$$\Longrightarrow \quad \int_0^t \|z'(s) + \xi(s)\|^2 \,\mathrm{d}s = 0$$

Third step: LOWER energy estimate and conclusion

UPPER energy estimate

If $\partial_z^{\mathrm{fr}}\mathcal{E}$ **strongly-weakly closed**, then we have **energy inequality**

$$\frac{1}{2}\int_0^t \|z'(s)\|^2\,\mathrm{d}s + \frac{1}{2}\int_0^t \|\xi(s)\|^2\mathrm{d}s + \mathcal{E}(z(t)) \le \mathcal{E}(z_0), \quad \xi(s) \in \partial_z^{\mathrm{fr}}\mathcal{E}(z(s))$$

♣ **IF** $\partial_z^{\mathrm{fr}}\mathcal{E}$ fulfils the **chain rule**

$$\begin{cases} z \in H^1(0,T;\mathcal{H}), \\ \xi \in L^2(0,T;\mathcal{H}), \\ \xi(t) \in \partial_z^{\mathrm{fr}}\mathcal{E}(z(t)) \text{ a.e. in } (0,T) \end{cases} \Rightarrow \frac{d}{dt}\mathcal{E}(z(t)) = \langle \xi(t), z'(t)\rangle \text{ a.e. in } (0,T)$$

then

$$\frac{1}{2}\int_0^t \|z'(s)\|^2\,\mathrm{d}s + \frac{1}{2}\int_0^t \|\xi(s)\|^2\mathrm{d}s + \mathcal{E}(z(t)) \le \mathcal{E}(z_0)$$
$$= \mathcal{E}(z(t)) - \int_0^t \langle \xi(s), z'(s)\rangle\,\mathrm{d}s$$

whence

$$\Longrightarrow \qquad z'(t) + \xi(t) \ni 0, \quad \xi(t) \in \partial_z^{\mathrm{fr}}\mathcal{E}(z(t)) \quad \text{for a.a. } t \in (0,T).$$

Third step: LOWER energy estimate and conclusion

UPPER energy estimate

If ${\partial_z}^{fr}\mathcal{E}$ **strongly-weakly closed**, then we have **energy inequality**

$$\frac{1}{2}\int_0^t \|z'(s)\|^2\,\mathrm{d}s + \frac{1}{2}\int_0^t \|\xi(s)\|^2\mathrm{d}s + \mathcal{E}(z(t)) \le \mathcal{E}(z_0), \quad \xi(s) \in {\partial_z}^{fr}\mathcal{E}(z(s))$$

♣ **IF** ${\partial_z}^{fr}\mathcal{E}$ fulfils the **chain rule**

$$\begin{cases} z \in H^1(0,T;\mathcal{H}), \\ \xi \in L^2(0,T;\mathcal{H}), \\ \xi(t) \in {\partial_z}^{fr}\mathcal{E}(z(t)) \text{ a.e. in } (0,T) \end{cases} \Rightarrow \frac{d}{dt}\mathcal{E}(z(t)) = \langle \xi(t), z'(t)\rangle \text{ a.e. in } (0,T)$$

then

$$\frac{1}{2}\int_0^t \|z'(s)\|^2\,\mathrm{d}s + \frac{1}{2}\int_0^t \|\xi(s)\|^2\mathrm{d}s + \mathcal{E}(z(t)) \le \mathcal{E}(z_0)$$

$$= \mathcal{E}(z(t)) - \int_0^t \langle \xi(s), z'(s)\rangle\,\mathrm{d}s$$

then z solves the gradient flow eq. and fulfils the **energy identity**

$$\frac{1}{2}\int_0^t \|z'(s)\|^2\,\mathrm{d}s + \frac{1}{2}\int_0^t \|\xi(s)\|^2\mathrm{d}s + \mathcal{E}(z(t)) = \mathcal{E}(z_0)$$

An existence and approximation result (I)

Theorem 1 [R., Savaré ESAIM-COCV'06]

Assume

- $\mathcal{E}$ is **coercive**,
- $\partial_z{}^{\mathsf{fr}}\mathcal{E}$ is **strongly-weakly closed**,
- $\partial_z{}^{\mathsf{fr}}\mathcal{E}$ complies with the **chain rule**.

An existence and approximation result (I)

Theorem 1 [R., Savaré ESAIM-COCV'06]

Assume

- $\mathcal{E}$ is **coercive**,
- $\partial_z^{\mathrm{fr}}\mathcal{E}$ is **strongly-weakly closed**,
- $\partial_z^{\mathrm{fr}}\mathcal{E}$ complies with the **chain rule**.

Then, the approximate solutions converge to a function $z \in H^1(0,T;\mathcal{H})$ which **solves** the Cauchy problem

$$z'(t) + \partial_z^{\mathrm{fr}}(z(t)) \ni 0 \quad \text{for a.a. } t \in (0,T), \qquad z(0) = z_0.$$

An existence and approximation result (I)

Theorem 1 [R., Savaré ESAIM-COCV'06]

Assume

- $\mathcal{E}$ is **coercive**,
- $\partial_z{}^{\mathbf{fr}}\mathcal{E}$ is **strongly-weakly closed**,
- $\partial_z{}^{\mathbf{fr}}\mathcal{E}$ complies with the **chain rule**.

Then, the approximate solutions converge to a function $z \in H^1(0,T;\mathcal{H})$ which **solves** the Cauchy problem

$$z'(t) + \partial_z{}^{\mathbf{fr}}(z(t)) \ni 0 \quad \text{for a.a. } t \in (0,T), \qquad z(0) = z_0.$$

Moreover, z fulfils the **energy identity** for all $0 \le s \le t \le T$

$$\frac{1}{2}\int_s^t \|z'(r)\|^2\,\mathrm{d}r + \underbrace{\frac{1}{2}\int_s^t \|\xi(r)\|^2\mathrm{d}r}_{=\frac{1}{2}\int_s^t \|z'(r)\|^2\mathrm{d}r} + \mathcal{E}(z(t)) = \mathcal{E}(z(s))$$

An existence and approximation result (I)

Theorem 1 [R., Savaré ESAIM-COCV'06]

Assume

- $\mathcal{E}$ is **coercive**,
- $\boldsymbol{\partial}_z{}^{\mathbf{fr}}\mathcal{E}$ is **strongly-weakly closed**,
- $\boldsymbol{\partial}_z{}^{\mathbf{fr}}\mathcal{E}$ complies with the **chain rule**.

Then, the approximate solutions converge to a function $z \in H^1(0,T;\mathcal{H})$ which **solves** the Cauchy problem

$$z'(t) + \boldsymbol{\partial}_z{}^{\mathbf{fr}}(z(t)) \ni 0 \quad \text{for a.a. } t \in (0,T), \qquad z(0) = z_0.$$

Moreover, z fulfils the **energy identity** for all $0 \le s \le t \le T$

$$\int_s^t \|z'(r)\|^2 \,\mathrm{d}r + \mathcal{E}(z(t)) = \mathcal{E}(z(s))$$

An existence and approximation result (I)

Theorem 1 [R., Savaré ESAIM-COCV'06]

Assume

- $\mathcal{E}$ is **coercive**,
- $\boldsymbol{\partial_z}^{\mathbf{fr}}\mathcal{E}$ is **strongly-weakly closed**,
- $\boldsymbol{\partial_z}^{\mathbf{fr}}\mathcal{E}$ complies with the **chain rule**.

Then, the approximate solutions converge to a function $z \in H^1(0, T; \mathcal{H})$ solving

$$z'(t) + \boldsymbol{\partial_z}^{\mathbf{fr}}(z(t)) \ni 0 \ \text{ for a.a. } t \in (0, T), \quad z(0) = z_0.$$

Moreover, z fulfils the **energy identity** for all $0 \leq s \leq t \leq T$

$$\int_s^t \|z'(r)\|^2 \, \mathrm{d}r + \mathcal{E}(z(t)) = \mathcal{E}(z(s))$$

Closedness and chain rule of $\boldsymbol{\partial_z}^{\mathbf{fr}}\mathcal{E}$

satisfied for **dominated concave perturbations of convex** functionals, viz.

$$\mathcal{E}(z) = \mathcal{E}_1(z) + \mathcal{E}_2(z) \ \text{ with } \begin{cases} \mathcal{E}_1 \text{ convex} \\ \mathcal{E}_2 \text{ concave} \\ \|\partial\mathcal{E}_2(z)\| \leq c\|\partial\mathcal{E}_1(z)\| + C \end{cases}$$

A more general viewpoint

Replace $\boldsymbol{\partial_z}^{\mathbf{fr}}\mathcal{E}$ with a GENERAL subdifferential notion $\partial_z\mathcal{E}$... which properties do we need on $\partial_z\mathcal{E}$?

A more general viewpoint

Proof of existence – revisited

Step 0: incremental minimization

$$Z_\tau^n \in \mathrm{Argmin}_{z\in\mathcal{H}} \left\{ \frac{1}{2\tau} \|z - Z_\tau^{n-1}\|^2 + \mathcal{E}(z) \right\}, \quad Z_\tau^0 := z_0$$

A more general viewpoint

Proof of existence – revisited

Step 0: incremental minimization: OK if $\mathcal{E}$ is **coercive**

$$Z_\tau^n \in \mathrm{Argmin}_{z\in\mathcal{H}} \left\{ \frac{1}{2\tau}\|z - Z_\tau^{n-1}\|^2 + \mathcal{E}(z) \right\}, \quad Z_\tau^0 := z_0$$

A more general viewpoint

Proof of existence – revisited

Step 0: incremental minimization: OK if $\mathcal{E}$ is **coercive**

$$Z_\tau^n \in \mathrm{Argmin}_{z\in\mathcal{H}} \left\{ \frac{1}{2\tau}\|z - Z_\tau^{n-1}\|^2 + \mathcal{E}(z) \right\}, \quad Z_\tau^0 := z_0$$

Step 1: approximate energy inequality

$$\frac{1}{2}\int_0^t \|\widehat{Z}_\tau'(s)\|^2 \,\mathrm{d}s + \frac{1}{2}\int_0^t \|\partial_z{}^{\mathbf{fr}}\mathcal{E}(\widetilde{Z}_\tau(s))\|^2 \mathrm{d}s + \mathcal{E}(\widetilde{Z}_\tau(t)) \le \mathcal{E}(z_0)$$

A more general viewpoint

Proof of existence – revisited

Step 0: incremental minimization: OK if $\mathcal{E}$ is **coercive**

$$Z_\tau^n \in \mathrm{Argmin}_{z\in\mathcal{H}} \left\{ \frac{1}{2\tau}\|z - Z_\tau^{n-1}\|^2 + \mathcal{E}(z) \right\}, \quad Z_\tau^0 := z_0$$

Step 1: approximate energy inequality: depends on **Euler equation** $\frac{\widetilde{Z}_\tau(t)-Z_\tau^{n-1}}{(t-t_{n-1})} + \boldsymbol{\partial_z}^{\mathbf{fr}}\mathcal{E}(\widetilde{Z}_\tau(t)) \ni 0$

$$\frac{1}{2}\int_0^t \|\widehat{Z}_\tau'(s)\|^2\,\mathrm{d}s + \frac{1}{2}\int_0^t \|\boldsymbol{\partial_z}^{\mathbf{fr}}\mathcal{E}(\widetilde{Z}_\tau(s))\|^2\mathrm{d}s + \mathcal{E}(\widetilde{Z}_\tau(t)) \leq \mathcal{E}(z_0)$$

A more general viewpoint

Proof of existence – revisited

Step 0: incremental minimization: OK if $\mathcal{E}$ is **coercive**

$$Z_\tau^n \in \mathrm{Argmin}_{z \in \mathcal{H}} \left\{ \frac{1}{2\tau} \|z - Z_\tau^{n-1}\|^2 + \mathcal{E}(z) \right\}, \quad Z_\tau^0 := z_0$$

Step 1: approximate energy inequality: depends on **Euler equation** $\frac{\widetilde{Z}_\tau(t) - Z_\tau^{n-1}}{(t - t_{n-1})} + \partial_z{}^{\mathrm{fr}} \mathcal{E}(\widetilde{Z}_\tau(t)) \ni 0$

$$\frac{1}{2} \int_0^t \|\widehat{Z}_\tau'(s)\|^2 \,\mathrm{d}s + \frac{1}{2} \int_0^t \|\partial_z{}^{\mathrm{fr}} \mathcal{E}(\widetilde{Z}_\tau(s))\|^2 \mathrm{d}s + \mathcal{E}(\widetilde{Z}_\tau(t)) \leq \mathcal{E}(z_0)$$

Step 2: UPPER energy estimate

$$\frac{1}{2} \int_0^t \|z'(s)\|^2 \,\mathrm{d}s + \frac{1}{2} \int_0^t \|\partial_z{}^{\mathrm{fr}} \mathcal{E}(s)\|^2 \mathrm{d}s + \mathcal{E}(z(t)) \leq \mathcal{E}(z_0)$$

A more general viewpoint

Proof of existence – revisited

Step 0: incremental minimization: OK if $\mathcal{E}$ is **coercive**

$$Z_\tau^n \in \mathrm{Argmin}_{z\in\mathcal{H}} \left\{ \frac{1}{2\tau}\|z - Z_\tau^{n-1}\|^2 + \mathcal{E}(z) \right\}, \quad Z_\tau^0 := z_0$$

Step 1: approximate energy inequality: depends on **Euler equation** $\frac{\widetilde{Z}_\tau(t) - Z_\tau^{n-1}}{(t-t_{n-1})} + \boldsymbol{\partial_z}^{\mathbf{fr}}\mathcal{E}(\widetilde{Z}_\tau(t)) \ni 0$

$$\frac{1}{2}\int_0^t \|\widehat{Z}_\tau'(s)\|^2 \, \mathrm{d}s + \frac{1}{2}\int_0^t \|\boldsymbol{\partial_z}^{\mathbf{fr}}\mathcal{E}(\widetilde{Z}_\tau(s))\|^2 \mathrm{d}s + \mathcal{E}(\widetilde{Z}_\tau(t)) \leq \mathcal{E}(z_0)$$

Step 2: UPPER energy estimate: holds if $\boldsymbol{\partial_z}^{\mathbf{fr}}\mathcal{E}$ **strongly-weakly closed**

$$\frac{1}{2}\int_0^t \|z'(s)\|^2 \, \mathrm{d}s + \frac{1}{2}\int_0^t \|\boldsymbol{\partial_z}^{\mathbf{fr}}\mathcal{E}(s)\|^2 \mathrm{d}s + \mathcal{E}(z(t)) \leq \mathcal{E}(z_0)$$

A more general viewpoint

Proof of existence – revisited

Step 0: incremental minimization: OK if $\mathcal{E}$ is **coercive**

$$Z_\tau^n \in \mathrm{Argmin}_{z\in\mathcal{H}} \left\{ \frac{1}{2\tau}\|z - Z_\tau^{n-1}\|^2 + \mathcal{E}(z) \right\}, \quad Z_\tau^0 := z_0$$

Step 1: approximate energy inequality: depends on **Euler equation** $\frac{\widetilde{Z}_\tau(t) - Z_\tau^{n-1}}{(t-t_{n-1})} + \boldsymbol{\partial_z}^{\mathbf{fr}}\mathcal{E}(\widetilde{Z}_\tau(t)) \ni 0$

$$\frac{1}{2}\int_0^t \|\widehat{Z}_\tau'(s)\|^2\,\mathrm{d}s + \frac{1}{2}\int_0^t \|\boldsymbol{\partial_z}^{\mathbf{fr}}\mathcal{E}(\widetilde{Z}_\tau(s))\|^2\mathrm{d}s + \mathcal{E}(\widetilde{Z}_\tau(t)) \leq \mathcal{E}(z_0)$$

Step 2: UPPER energy estimate: holds if $\boldsymbol{\partial_z}^{\mathbf{fr}}\mathcal{E}$ **strongly-weakly closed**

$$\frac{1}{2}\int_0^t \|z'(s)\|^2\,\mathrm{d}s + \frac{1}{2}\int_0^t \|\boldsymbol{\partial_z}^{\mathbf{fr}}\mathcal{E}(s)\|^2\mathrm{d}s + \mathcal{E}(z(t)) \leq \mathcal{E}(z_0)$$

Step 3: LOWER energy estimate

$$\mathcal{E}(z_0) = \mathcal{E}(z(t)) - \int_0^t \langle \boldsymbol{\partial_z}^{\mathbf{fr}}\mathcal{E}(s), z'(s)\rangle\,\mathrm{d}s \leq \frac{1}{2}\int_0^t |z'(s)\|^2\,\mathrm{d}s + \frac{1}{2}\int_0^t \|\boldsymbol{\partial_z}^{\mathbf{fr}}\mathcal{E}(s)\|^2\mathrm{d}s + \mathcal{E}(z(t))$$

A more general viewpoint

Proof of existence – revisited

Step 0: incremental minimization: OK if $\mathcal{E}$ is **coercive**

$$\mathrm{Z}_\tau^n \in \mathrm{Argmin}_{z\in\mathcal{H}} \left\{ \frac{1}{2\tau}\|z - \mathrm{Z}_\tau^{n-1}\|^2 + \mathcal{E}(z) \right\}, \quad \mathrm{Z}_\tau^0 := z_0$$

Step 1: approximate energy inequality: depends on **Euler equation** $\frac{\widetilde{\mathrm{Z}}_\tau(t)-\mathrm{Z}_\tau^{n-1}}{(t-t_{n-1})} + \boldsymbol{\partial_z}^{\mathbf{fr}}\mathcal{E}(\widetilde{\mathrm{Z}}_\tau(t)) \ni 0$

$$\frac{1}{2}\int_0^t \|\widehat{\mathrm{Z}}_\tau'(s)\|^2\,\mathrm{d}s + \frac{1}{2}\int_0^t \|\boldsymbol{\partial_z}^{\mathbf{fr}}\mathcal{E}(\widetilde{\mathrm{Z}}_\tau(s))\|^2\mathrm{d}s + \mathcal{E}(\widetilde{\mathrm{Z}}_\tau(t)) \leq \mathcal{E}(z_0)$$

Step 2: UPPER energy estimate: holds if $\boldsymbol{\partial_z}^{\mathbf{fr}}\mathcal{E}$ **strongly-weakly closed**

$$\frac{1}{2}\int_0^t \|z'(s)\|^2\,\mathrm{d}s + \frac{1}{2}\int_0^t \|\boldsymbol{\partial_z}^{\mathbf{fr}}\mathcal{E}(s)\|^2\mathrm{d}s + \mathcal{E}(z(t)) \leq \mathcal{E}(z_0)$$

Step 3: LOWER energy estimate:

$$\mathcal{E}(z_0) = \mathcal{E}(z(t)) - \int_0^t \langle \boldsymbol{\partial_z}^{\mathbf{fr}}\mathcal{E}(s), z'(s)\rangle\,\mathrm{d}s \leq \frac{1}{2}\int_0^t \|z'(s)\|^2\,\mathrm{d}s + \frac{1}{2}\int_0^t \|\boldsymbol{\partial_z}^{\mathbf{fr}}\mathcal{E}(s)\|^2\mathrm{d}s + \mathcal{E}(z(t))$$

holds if $\boldsymbol{\partial_z}^{\mathbf{fr}}\mathcal{E}$ **fulfils chain rule** $\frac{d}{dt}\mathcal{E}(z(t)) = \langle \boldsymbol{\partial_z}^{\mathbf{fr}}\mathcal{E}(z(t)), z'(t)\rangle$

A general subdifferential notion

From now on, **back to Banach** spaces:

$$\mathcal{E} : B \to (-\infty, +\infty] \text{ such that } z \mapsto \mathcal{E}(z) \text{ is } \textbf{coercive}$$

- We work with $\partial_z \mathcal{E} : B \rightrightarrows B^*$ fulfilling

Euler equation for minimimum problems

$$\bar{Z} \in \mathrm{Argmin}_{z \in B} \left\{ \frac{1}{2\tau} \|z - Z_o\|^2 + \mathcal{E}(v) \right\} \Rightarrow \quad \frac{\bar{Z} - Z_o}{\tau} + \partial_z \mathcal{E}(\bar{Z}) \ni 0$$

viz.

$$\exists \xi \in \partial_z \mathcal{E}(\bar{Z}) \text{ s.t. } \frac{\bar{Z} - Z_o}{\tau} + \xi = 0$$

strong-weak closedness

$$z_n \to z, \quad \xi_n \rightharpoonup \xi, \quad \sup_n \mathcal{E}(z_n) < +\infty, \quad \xi_n \in \partial_z \mathcal{E}(z_n) \ \forall n \in \mathbb{N} \quad \Rightarrow \quad \xi \in \partial_z \mathcal{E}(z)$$

chain rule inequality

$$-\frac{d}{dt} \mathcal{E}(z(t)) \leq -\langle \xi(t), z'(t) \rangle \quad \text{for all } \xi(t) \in \partial_z \mathcal{E}(z(t)).$$

A general subdifferential notion

Back to Banach spaces:

$$\mathcal{E} : B \to (-\infty, +\infty] \text{ such that } z \mapsto \mathcal{E}(z) \text{ **coercive**}$$

Conditions on $\partial_z \mathcal{E} : B \rightrightarrows B^*$

Euler equation for minimimum problems

strong-weak closedness

chain rule inequality

Examples

$\mathcal{E} \in \mathrm{C}^1(B)$	$\Rightarrow$	$\partial_z \mathcal{E}(z) = \mathrm{D}\mathcal{E}(z)$	**Fréchet diff.**	**OK**
$\mathcal{E} : B \to (-\infty, +\infty]$ **convex**	$\Rightarrow$	$\partial_z \mathcal{E}(z) = \partial_z{}^{\mathrm{co}} \mathcal{E}(z)$	**cvx subdif.**	**OK**
$\mathcal{E} : B \to (-\infty, +\infty]$ $\lambda-$**convex**	$\Rightarrow$	$\partial_z \mathcal{E}(z) = \partial_z{}^{\mathrm{fr}} \mathcal{E}(z)$	**Fréchet subdif.**	**OK**

A general subdifferential notion

Back to Banach spaces:

$$\mathcal{E} : B \to (-\infty, +\infty] \text{ such that } z \mapsto \mathcal{E}(z) \textbf{coercive}$$

Conditions on $\partial_z \mathcal{E} : B \rightrightarrows B^*$

Euler equation for minimimum problems

strong-weak closedness

chain rule inequality

Examples

♣ If

$$\mathcal{E}(z) := \min_{\eta \in \mathcal{F}} I(\eta, z) \text{ **marginal functional**}$$

then a suitable subdifferential notion is

$$\partial_z \mathcal{E}(z) = \boldsymbol{\partial_z}^{\mathbf{ma}} \mathcal{E}(z) = \{\mathrm{D}_z I(t, \eta, z) \ : \ \eta \in \mathrm{Argmin} I(t, \cdot, z)\} \quad \textbf{Marginal subdifferential}$$

General dissipation, smooth time-dependence

Dissipation potential

From $\Psi(v) = \frac{1}{2}\|v\|^2$, we pass to a **general**

$$\Psi : B \to [0, +\infty) \text{ l.s.c. \& convex}, \quad \Psi(0) = \min_{v \in B} \Psi(v) = 0,$$

$$\lim_{\|v\| \to \infty} \frac{\Psi(v)}{\|v\|} = +\infty, \quad \lim_{\|\xi\|_* \to \infty} \frac{\Psi^*(\xi)}{\|\xi\|_*} = +\infty$$

with $\Psi^* : B^* \to [0, +\infty)$ conjugate functional to Ψ

General dissipation, smooth time-dependence

Dissipation potential

From $\Psi(v) = \frac{1}{2}\|v\|^2$, we pass to a **general**

$$\Psi : B \to [0, +\infty) \text{ l.s.c. \& convex}, \quad \Psi(0) = \min_{v \in B} \Psi(v) = 0,$$

$$\lim_{\|v\| \to \infty} \frac{\Psi(v)}{\|v\|} = +\infty, \qquad \lim_{\|\xi\|_* \to \infty} \frac{\Psi^*(\xi)}{\|\xi\|_*} = +\infty$$

with $\Psi^* : B^* \to [0, +\infty)$ conjugate functional to Ψ

Back to a *time-dependent* energy $\boxed{\mathcal{E} : [0, T] \times B \to (-\infty, +\infty+]}$, such that

$$z \mapsto \mathcal{E}(t, z) \text{ is lower semicontinuous and } \textbf{coercive}$$

General dissipation, smooth time-dependence

Dissipation potential

From $\Psi(v) = \frac{1}{2}\|v\|^2$, we pass to a **general**

$$\Psi : B \to [0, +\infty) \text{ l.s.c. \& convex}, \quad \Psi(0) = \min_{v \in B} \Psi(v) = 0,$$

$$\lim_{\|v\| \to \infty} \frac{\Psi(v)}{\|v\|} = +\infty, \qquad \lim_{\|\xi\|_* \to \infty} \frac{\Psi^*(\xi)}{\|\xi\|_*} = +\infty$$

with $\Psi^* : B^* \to [0, +\infty)$ conjugate functional to Ψ

Back to a *time-dependent* energy $\boxed{\mathcal{E} : [0, T] \times B \to (-\infty, +\infty+]}$, such that

$$z \mapsto \mathcal{E}(t, z) \text{ is lower semicontinuous and } \textbf{coercive}$$

For the moment: $\boxed{t \mapsto \mathcal{E}(t, z) \text{ is } \textbf{differentiable}}$

General dissipation, smooth time-dependence

Dissipation potential

From $\Psi(v) = \frac{1}{2}\|v\|^2$, we pass to a **general**

$$\Psi : B \to [0, +\infty) \text{ l.s.c. \& convex}, \quad \Psi(0) = \min_{v \in B} \Psi(v) = 0,$$

$$\lim_{\|v\| \to \infty} \frac{\Psi(v)}{\|v\|} = +\infty, \qquad \lim_{\|\xi\|_* \to \infty} \frac{\Psi^*(\xi)}{\|\xi\|_*} = +\infty$$

with $\Psi^* : B^* \to [0, +\infty)$ conjugate functional to Ψ

Back to a *time-dependent* energy $\boxed{\mathcal{E} : [0, T] \times B \to (-\infty, +\infty+]}$, such that

$$z \mapsto \mathcal{E}(t, z) \text{ is lower semicontinuous and } \textbf{coercive}$$

For the moment: $\boxed{t \mapsto \mathcal{E}(t, z) \text{ is } \textbf{differentiable}}$

Work with $\partial_z \mathcal{E}(t, \cdot) : B \rightrightarrows B^*$ fulfilling

$$\left.\begin{array}{r} \textbf{Euler equation for minimimum problems} \\ \textbf{strong-weak closedness} \\ \textbf{chain rule inequality} \end{array}\right\}$$

General dissipation, smooth time-dependence

Dissipation potential

From $\Psi(v) = \frac{1}{2}\|v\|^2$, we pass to a **general**

$$\Psi : B \to [0,+\infty) \text{ l.s.c. \& convex}, \quad \Psi(0) = \min_{v\in B} \Psi(v) = 0,$$

$$\lim_{\|v\|\to\infty} \frac{\Psi(v)}{\|v\|} = +\infty, \qquad \lim_{\|\xi\|_*\to\infty} \frac{\Psi^*(\xi)}{\|\xi\|_*} = +\infty$$

with $\Psi^* : B^* \to [0,+\infty)$ conjugate functional to Ψ

Back to a *time-dependent* energy $\boxed{\mathcal{E} : [0,T]\times B \to (-\infty,+\infty+]}$, such that

$$z \mapsto \mathcal{E}(t,z) \text{ is lower semicontinuous and } \textbf{coercive}$$

For the moment: $\boxed{t \mapsto \mathcal{E}(t,z) \text{ is } \textbf{differentiable}}$
Work with $\partial_z\mathcal{E}(t,\cdot) : B \rightrightarrows B^*$ fulfilling

Euler equation for minimimum problems
strong-weak closedness
chain rule inequality
} in the **doubly nonlinear** setting

Step 0

Cauchy Problem:

$$\begin{cases} \partial\Psi(z'(t)) + \partial_z\mathcal{E}(t,z(t)) \ni 0 \quad \text{in } B', \quad t \in (0,T), \\ z(0) = z_0 \end{cases}$$

Step 0: time-incremental minimization

Fix time-step $\tau > 0$ and recursively find $Z_\tau^0 := z_0,\, Z_\tau^1,\, \ldots,\, Z_\tau^N$:

$$Z_\tau^n \in \mathrm{Argmin}_{z\in B} \left\{ \frac{1}{2\tau}\|z - Z_\tau^{n-1}\|^2 + \mathcal{E}(z) \right\},$$

Step 0

Cauchy Problem:

$$\begin{cases} \partial\Psi(z'(t)) + \partial_z \mathcal{E}(t, z(t)) \ni 0 \quad \text{in } B', \quad t \in (0, T), \\ z(0) = z_0 \end{cases}$$

Step 0: time-incremental minimization

Fix time-step $\tau > 0$ and recursively find $Z_\tau^0 := z_0, Z_\tau^1, \ldots, Z_\tau^N$:

$$Z_\tau^n \in \mathrm{Argmin}_{z \in B} \left\{ \tau\Psi\left(\frac{z - Z_\tau^{n-1}}{\tau}\right) + \mathcal{E}(t_n, z) \right\},$$

♣ **OK**, because $\mathcal{E}(t, \cdot)$ is **coercive**

Step 0

Cauchy Problem:

$$\begin{cases} \partial\Psi(z'(t)) + \partial_z\mathcal{E}(t, z(t)) \ni 0 & \text{in } B', \quad t \in (0, T), \\ z(0) = z_0 \end{cases}$$

Step 0: time-incremental minimization

Fix time-step $\tau > 0$ and recursively find $Z_\tau^0 := z_0, Z_\tau^1, \ldots, Z_\tau^N$:

$$Z_\tau^n \in \mathrm{Argmin}_{z \in B} \left\{ \tau\Psi\left(\frac{z - Z_\tau^{n-1}}{\tau}\right) + \mathcal{E}(t_n, z) \right\},$$

♣ **OK**, because $\mathcal{E}(t, \cdot)$ is **coercive**
$\partial_z\mathcal{E}$ fulfils **Euler equation for minimum problems** $\Rightarrow$

$$\partial\Psi\left(\frac{Z_\tau^n - Z_\tau^{n-1}}{\tau}\right) + \partial_z\mathcal{E}(t_n, Z_\tau^n) \ni 0$$

Step 1

Approximate solutions:

$\{\overline{Z}_\tau\}$ piecewise constant interpolant;

$\{\widehat{Z}_\tau\}$ piecewise linear interpolant;

$\{\widetilde{Z}_\tau\}$ *variational* interpolant:

$$\widetilde{Z}_\tau(t) \in \mathrm{Argmin}_{Z\in B} \left\{ (t-t_{n-1})\Psi\left(\frac{z - Z_\tau^{n-1}}{t-t_{n-1}}\right) + \mathcal{E}(t,z) \right\} \text{ for } t \in (t_{n-1}, t_n]$$

Step 1: approximate energy inequality

$$\int_0^t \Psi(\widehat{Z}_\tau'(s))\,\mathrm{d}s + \int_0^t \Psi^*(-\widetilde{\xi}_\tau(s))\,\mathrm{d}s + \mathcal{E}(t,\widetilde{Z}_\tau(t)) \leq \mathcal{E}(0,z_0) + \int_0^t \partial_t \mathcal{E}(s,\widetilde{Z}_\tau(s))\,\mathrm{d}s$$

with for a.a. $s \in (0,t)$:

$$\widetilde{\xi}_\tau(s) \in \partial_z \mathcal{E}(s,\widetilde{Z}_\tau(s)), \quad \partial\Psi\left(\frac{\widetilde{Z}_\tau(s) - Z_\tau^{n-1}}{t-t_{n-1}}\right) + \widetilde{\xi}_\tau(s) \ni 0.$$

Step 1

Step 1: approximate energy inequality

$$\int_0^t \Psi(\widehat{Z}_\tau'(s))\,\mathrm{d}s + \int_0^t \Psi^*(-\widetilde{\xi}_\tau(s))\,\mathrm{d}s + \mathcal{E}(t,\widetilde{Z}_\tau(t)) \leq \mathcal{E}(0,z_0) + \int_0^t \partial_t\mathcal{E}(s,\widetilde{Z}_\tau(s))\,\mathrm{d}s$$

with for a.a. $s \in (0,t)$:

$$\widetilde{\xi}_\tau(s) \in \partial_z\mathcal{E}(s,\widetilde{Z}_\tau(s)), \quad \partial\Psi\left(\frac{\widetilde{Z}_\tau(s) - Z_\tau^{n-1}}{t - t_{n-1}}\right) + \widetilde{\xi}_\tau(s) \ni 0.$$

$\Rightarrow$ **A priori estimates** for $(\overline{Z}_\tau)_\tau$, $(\widehat{Z}_\tau)_\tau$, $(\widetilde{Z}_\tau)_\tau$, $(\widetilde{\xi}_\tau)_\tau$. Since Ψ & Ψ^* have **superlinear growth**, we have compactness:

$$\begin{array}{ll} \overline{Z}_\tau,\ \widehat{Z}_\tau,\ \widetilde{Z}_\tau \to z & \text{in } L^\infty(0,T;B), \\ \widehat{Z}_\tau' \rightharpoonup z' & \text{in } L^1(0,T;B), \\ \widetilde{\xi}_\tau \rightharpoonup \xi & \text{in } L^1(0,T;B^*). \end{array}$$

Step 2

Step 2: UPPER energy estimate

Passage to the limit

$$\liminf_{\tau\downarrow 0}\int_0^t \Psi(\widehat{Z}_\tau'(s))\,\mathrm{d}s + \liminf_{\tau\downarrow 0}\int_0^t \Psi^*(-\widetilde{\xi}_\tau(s))\,\mathrm{d}s + \liminf_{\tau\downarrow 0}\mathcal{E}(t,\widetilde{Z}_\tau(t))$$

$$\leq \mathcal{E}(0,z_0) + \limsup_{\tau\downarrow 0}\int_0^t \partial_t\mathcal{E}(s,\widetilde{Z}_\tau(s))\,\mathrm{d}s$$

$\downarrow$ (via **LOWER SEMICONTINUITY**)

$$\int_0^t \Psi(z'(s))\,\mathrm{d}s + \int_0^t \Psi^*(-\xi(s))\,\mathrm{d}s + \mathcal{E}(t,z(t)) \leq \mathcal{E}(0,z_0) + \int_0^t \partial_t\mathcal{E}(s,z(s))\,\mathrm{d}s$$

Step 2

Step 2: UPPER energy estimate

Passage to the limit

$$\liminf_{\tau\downarrow 0}\int_0^t \Psi(\widehat{Z}_\tau'(s))\,\mathrm{d}s + \liminf_{\tau\downarrow 0}\int_0^t \Psi^*(-\widetilde{\xi}_\tau(s))\,\mathrm{d}s + \liminf_{\tau\downarrow 0}\mathcal{E}(t,\widetilde{Z}_\tau(t))$$
$$\leq \mathcal{E}(0,z_0) + \limsup_{\tau\downarrow 0}\int_0^t \partial_t\mathcal{E}(s,\widetilde{Z}_\tau(s))\,\mathrm{d}s$$

$\downarrow$ (via **LOWER SEMICONTINUITY**)

$$\int_0^t \Psi(z'(s))\,\mathrm{d}s + \int_0^t \Psi^*(-\xi(s))\,\mathrm{d}s + \mathcal{E}(t,z(t)) \leq \mathcal{E}(0,z_0) + \int_0^t \partial_t\mathcal{E}(s,z(s))\,\mathrm{d}s$$

with $\boxed{\xi(s)\in\partial_z\mathcal{E}(s,z(s)) \text{ for a.a. } s\in(0,t)}$ **IF** $(\partial_z\mathcal{E},\partial_t\mathcal{E})$ fulfils the **strong-weak closedness** property

$$z_n\to z,\quad \xi_n\rightharpoonup\xi,\quad \sup_n \mathcal{E}(t_n,z_n)<+\infty,\quad \xi_n\in\partial_z\mathcal{E}(t,z_n)\ \forall n\in\mathbb{N}$$
$$\Rightarrow\quad \xi\in\partial_z\mathcal{E}(t,z),\quad \limsup_{n\to\infty}\partial_t\mathcal{E}(t,z_n)\leq\partial_t\mathcal{E}(t,z)$$

Step 3

Step 3: LOWER energy estimate and conclusion

IF $\partial_z \mathcal{E}$ fulfils the **chain rule inequality**

$$\begin{cases} z \in W^{1,1}(0,T;B), \quad \int_0^T \Psi(z'(t))\,\mathrm{d}t < +\infty, \\ \xi \in L^1(0,T;B^*), \quad \int_0^T \Psi^*(\xi(t))\,\mathrm{d}t < +\infty, \\ \xi(t) \in \partial_z \mathcal{E}(t,z(t)) \text{ a.e. in } (0,T) \end{cases}$$

$$\Rightarrow \quad -\frac{d}{dt}\mathcal{E}(t,z(t)) + \partial_t \mathcal{E}(t,z(t)) \leq -\langle \xi(t), z'(t) \rangle \text{ a.e. in } (0,T)$$

Step 3

Step 3: LOWER energy estimate and conclusion

IF $\partial_z \mathcal{E}$ fulfils the **chain rule inequality**

$$\begin{cases} z \in W^{1,1}(0,T;B), \quad \int_0^T \Psi(z'(t))\,\mathrm{d}t < +\infty, \\ \xi \in L^1(0,T;B^*), \quad \int_0^T \Psi^*(\xi(t))\,\mathrm{d}t < +\infty, \\ \xi(t) \in \partial_z \mathcal{E}(t,z(t)) \text{ a.e. in } (0,T) \end{cases}$$

$$\Rightarrow \quad -\frac{d}{dt}\mathcal{E}(t,z(t)) + \partial_t \mathcal{E}(t,z(t)) \leq -\langle \xi(t), z'(t)\rangle \text{ a.e. in } (0,T)$$

then

$$\begin{aligned} \int_0^t \Psi(z'(s))\,\mathrm{d}s \quad & + \int_0^t \Psi^*(-\xi(s))\,\mathrm{d}s + \mathcal{E}(t,z(t)) \\ & \leq \mathcal{E}(0,z_0) + \int_0^t \partial_t \mathcal{E}(s,z(s))\,\mathrm{d}s \\ & \leq \mathcal{E}(t,z(t)) - \int_0^t \langle \xi(s), z'(s)\rangle\,\mathrm{d}s \\ & \leq \int_0^t \Psi(z'(s))\,\mathrm{d}s + \int_0^t \Psi^*(-\xi(s))\,\mathrm{d}s + \mathcal{E}(t,z(t)) \end{aligned}$$

Step 3

Step 3: LOWER energy estimate and conclusion

IF $\partial_z \mathcal{E}$ fulfils the **chain rule inequality**

$$\begin{cases} z \in W^{1,1}(0,T;B), & \int_0^T \Psi(z'(t))\,\mathrm{d}t < +\infty, \\ \xi \in L^1(0,T;B^*), & \int_0^T \Psi^*(\xi(t))\,\mathrm{d}t < +\infty, \\ \xi(t) \in \partial_z \mathcal{E}(t,z(t)) \text{ a.e. in } (0,T) \end{cases}$$

$$\Rightarrow \quad -\frac{d}{dt}\mathcal{E}(t,z(t)) + \partial_t \mathcal{E}(t,z(t)) \leq -\langle \xi(t), z'(t)\rangle \text{ a.e. in } (0,T)$$

then

$$\begin{aligned} \int_0^t \Psi(z'(s))\,\mathrm{d}s \quad & + \int_0^t \Psi^*(-\xi(s))\,\mathrm{d}s + \mathcal{E}(t,z(t)) \\ & \leq \mathcal{E}(0,z_0) + \int_0^t \partial_t \mathcal{E}(s,z(s))\,\mathrm{d}s \\ & \leq \mathcal{E}(t,z(t)) - \int_0^t \langle \xi(s), z'(s)\rangle\,\mathrm{d}s \\ & \leq \int_0^t \Psi(z'(s))\,\mathrm{d}s + \int_0^t \Psi^*(-\xi(s))\,\mathrm{d}s + \mathcal{E}(t,z(t)) \end{aligned}$$

Hence,

$$\int_0^t \Psi(z'(s))\,\mathrm{d}s + \int_0^t \Psi^*(-\xi(s))\,\mathrm{d}s - \int_0^t \langle -\xi(s), z'(s)\rangle\,\mathrm{d}s \leq 0$$

Step 3

Step 3: LOWER energy estimate and conclusion

IF $\partial_z\mathcal{E}$ fulfils the **chain rule inequality**

$$\begin{cases} z \in W^{1,1}(0,T;B), & \int_0^T \Psi(z'(t))\,\mathrm{d}t < +\infty, \\ \xi \in L^1(0,T;B^*), & \int_0^T \Psi^*(\xi(t))\,\mathrm{d}t < +\infty, \\ \xi(t) \in \partial_z\mathcal{E}(t,z(t)) \text{ a.e. in } (0,T) \end{cases}$$

$$\Rightarrow \quad -\frac{d}{dt}\mathcal{E}(t,z(t)) + \partial_t\mathcal{E}(t,z(t)) \le -\langle \xi(t), z'(t)\rangle \text{ a.e. in } (0,T)$$

then

$$\begin{aligned} \int_0^t \Psi(z'(s))\,\mathrm{d}s \quad & + \int_0^t \Psi^*(-\xi(s))\,\mathrm{d}s + \mathcal{E}(t,z(t)) \\ & \le \mathcal{E}(0,z_0) + \int_0^t \partial_t\mathcal{E}(s,z(s))\,\mathrm{d}s \\ & \le \mathcal{E}(t,z(t)) - \int_0^t \langle \xi(s), z'(s)\rangle\,\mathrm{d}s \\ & \le \int_0^t \Psi(z'(s))\,\mathrm{d}s + \int_0^t \Psi^*(-\xi(s))\,\mathrm{d}s + \mathcal{E}(t,z(t)) \end{aligned}$$

Hence,

$$\int_0^t \Psi(z'(s))\,\mathrm{d}s + \int_0^t \Psi^*(-\xi(s))\,\mathrm{d}s - \int_0^t \langle -\xi(s), z'(s)\rangle\,\mathrm{d}s = 0$$

Step 3

Step 3: LOWER energy estimate and conclusion

IF $\partial_z \mathcal{E}$ fulfils the **chain rule inequality**

$$\begin{cases} z \in W^{1,1}(0,T;B), \quad \int_0^T \Psi(z'(t))\,\mathrm{d}t < +\infty, \\ \xi \in L^1(0,T;B^*), \quad \int_0^T \Psi^*(\xi(t))\,\mathrm{d}t < +\infty, \\ \xi(t) \in \partial_z \mathcal{E}(t,z(t)) \text{ a.e. in } (0,T) \end{cases}$$

$$\Rightarrow \quad -\frac{d}{dt}\mathcal{E}(t,z(t)) + \partial_t \mathcal{E}(t,z(t)) \leq -\langle \xi(t), z'(t) \rangle \text{ a.e. in } (0,T)$$

then

$$\begin{aligned} \int_0^t \Psi(z'(s))\,\mathrm{d}s \quad & + \int_0^t \Psi^*(-\xi(s))\,\mathrm{d}s + \mathcal{E}(t,z(t)) \\ & \leq \mathcal{E}(0,z_0) + \int_0^t \partial_t \mathcal{E}(s,z(s))\,\mathrm{d}s \\ & \leq \mathcal{E}(t,z(t)) - \int_0^t \langle \xi(s), z'(s) \rangle \,\mathrm{d}s \\ & \leq \int_0^t \Psi(z'(s))\,\mathrm{d}s + \int_0^t \Psi^*(-\xi(s))\,\mathrm{d}s + \mathcal{E}(t,z(t)) \end{aligned}$$

Hence,

$$\Psi(z'(t)) + \Psi^*(-\xi(t)) - \langle -\xi(t), z'(t) \rangle = 0 \quad \text{for a.a. } t \in (0,T).$$

Step 3

Step 3: LOWER energy estimate and conclusion

IF $\partial_z \mathcal{E}$ fulfils the **chain rule inequality**

$$\begin{cases} z \in W^{1,1}(0,T;B), & \int_0^T \Psi(z'(t))\,\mathrm{d}t < +\infty, \\ \xi \in L^1(0,T;B^*), & \int_0^T \Psi^*(\xi(t))\,\mathrm{d}t < +\infty, \\ \xi(t) \in \partial_z \mathcal{E}(t,z(t)) \text{ a.e. in } (0,T) \end{cases}$$

$$\Rightarrow \quad -\frac{d}{dt}\mathcal{E}(t,z(t)) + \partial_t \mathcal{E}(t,z(t)) \leq -\langle \xi(t), z'(t)\rangle \text{ a.e. in } (0,T)$$

then

$$\begin{aligned} \int_0^t \Psi(z'(s))\,\mathrm{d}s \quad & + \int_0^t \Psi^*(-\xi(s))\,\mathrm{d}s + \mathcal{E}(t,z(t)) \\ & \leq \mathcal{E}(0,z_0) + \int_0^t \partial_t \mathcal{E}(s,z(s))\,\mathrm{d}s \\ & \leq \mathcal{E}(t,z(t)) - \int_0^t \langle \xi(s), z'(s)\rangle\,\mathrm{d}s \\ & \leq \int_0^t \Psi(z'(s))\,\mathrm{d}s + \int_0^t \Psi^*(-\xi(s))\,\mathrm{d}s + \mathcal{E}(t,z(t)) \end{aligned}$$

Hence,

$$-\xi(t) \in \partial\Psi(z'(t)), \quad \xi(t) \in \partial_z \mathcal{E}(t,z(t))$$

i.e. z is a solution of the Cauchy problem.

An existence and approximation result (II)

Theorem 2 [Mielke, R., Savaré2011]

Assume

- Ψ convex with **superlinear growth**,
- $\mathcal{E}(\cdot, z)$ **differentiable**
- $\mathcal{E}(t, \cdot)$ **coercive**,

and suppose that $\partial_z \mathcal{E}(t, \cdot)$ satisfies

Euler equation for minimimum problems
strong-weak closedness
chain rule inequality
} in the doubly nonlinear setting

An existence and approximation result (II)

Theorem 2 [Mielke, R., Savaré2011]

Assume

- Ψ convex with **superlinear growth**,
- $\mathcal{E}(\cdot, z)$ **differentiable**
- $\mathcal{E}(t, \cdot)$ **coercive**,

and suppose that $\partial_z \mathcal{E}(t, \cdot)$ satisfies

Euler equation for minimimum problems
strong-weak closedness
chain rule inequality
} in the doubly nonlinear setting

Then, the approximate solutions converge to a function $z \in H^1(0, T; \mathcal{H})$ which **solves** the Cauchy problem for (DNE), and fulfils the **energy identity** for all $0 \leq s \leq t \leq T$

$$\int_s^t \Psi(z'(r))\,\mathrm{d}r + \int_s^t \Psi^*(-\xi(r))\,\mathrm{d}r + \mathcal{E}(t,z(t)) = \mathcal{E}(s,z(s)) + \int_s^t \partial_t \mathcal{E}(r, z(r))\,\mathrm{d}r.$$

Nonsmooth time-dependence: a surrogate for $\partial_t \mathcal{E}$

♠ And what, if $t \mapsto \mathcal{E}(t, z)$ is **no longer smooth**? How to replace $\partial_t \mathcal{E}(t, z)$?

Nonsmooth time-dependence: a surrogate for $\partial_t \mathcal{E}$

♠ And what, if $t \mapsto \mathcal{E}(t,z)$ is **no longer smooth**? How to replace $\partial_t \mathcal{E}(t,z)$?

♣ Use a **generalized** partial time derivative $\boxed{\mathrm{P}_t \mathcal{E}}$, which must satisfy the **chain rule**

$$-\frac{d}{dt}\mathcal{E}(t,z(t)) + \mathrm{P}_t\mathcal{E}(t,z(t)) \leq -\langle \xi(t), z'(t)\rangle \text{ for all } \xi(t) \in \partial_z \mathcal{E}(t,z(t)), \text{ a.e. in } (0,T)$$

Nonsmooth time-dependence: a surrogate for $\partial_t \mathcal{E}$

♠ And what, if $t \mapsto \mathcal{E}(t,z)$ is **no longer smooth**? How to replace $\partial_t \mathcal{E}(t,z)$?

♣ Use a **generalized** partial time derivative $\boxed{\mathrm{P}_t \mathcal{E}}$, which must satisfy the **chain rule**

$$-\frac{d}{dt}\mathcal{E}(t,z(t)) + \mathrm{P}_t\mathcal{E}(t,z(t)) \leq -\langle \xi(t), z'(t) \rangle \text{ for all } \xi(t) \in \partial_z \mathcal{E}(t,z(t)), \text{ a.e. in } (0,T)$$

♣ Hence, $\mathrm{P}_t \mathcal{E}$ is *conditioned* to the subdifferential $\partial_z \mathcal{E}$, viz.

Nonsmooth time-dependence: a surrogate for $\partial_t \mathcal{E}$

♠ And what, if $t \mapsto \mathcal{E}(t, z)$ is **no longer smooth**? How to replace $\partial_t \mathcal{E}(t, z)$?

♣ Use a **generalized** partial time derivative $\boxed{\mathrm{P}_t \mathcal{E}}$, which must satisfy the **chain rule**

$$-\frac{d}{dt}\mathcal{E}(z(t)) + \mathrm{P}_t \mathcal{E}(t, z(t), \xi(t)) \leq -\langle \xi(t), z'(t) \rangle \text{ for all } \xi(t) \in \partial_z \mathcal{E}(t, z(t)), \text{ a.e. in } (0, T)$$

♣ Hence, $\mathrm{P}_t \mathcal{E}$ is *conditioned* to the subdifferential $\partial_z \mathcal{E}$, viz.

$$\mathrm{P}_t \mathcal{E} = \mathrm{P}_t \mathcal{E}(t, z, \xi), \quad \text{with } \xi \in \partial_z \mathcal{E}(t, z).$$

Nonsmooth time-dependence: a surrogate for $\partial_t \mathcal{E}$

♠ And what, if $t \mapsto \mathcal{E}(t,z)$ is **no longer smooth**? How to replace $\partial_t \mathcal{E}(t,z)$?

♣ Use a **generalized** partial time derivative $\boxed{\mathrm{P}_t \mathcal{E}}$, which must satisfy the **chain rule**

$-\frac{d}{dt}\mathcal{E}(z(t))+\mathrm{P}_t\mathcal{E}(t,z(t),\xi(t)) \leq -\langle \xi(t), z'(t)\rangle$ for all $\xi(t) \in \partial_z \mathcal{E}(t,z(t))$, a.e. in $(0,T)$

♣ Hence, $\mathrm{P}_t\mathcal{E}$ is *conditioned* to the subdifferential $\partial_z \mathcal{E}$, viz.

$$\mathrm{P}_t\mathcal{E} = \mathrm{P}_t\mathcal{E}(t,z,\xi), \quad \text{with } \xi \in \partial_z \mathcal{E}(t,z).$$

♢ For the existence theorem, impose that $\mathrm{P}_t\mathcal{E}$ complies with

strong-weak closedness

chain rule inequality

A material model with finite-strain elasticity

Variables

Elastic body in a bded domain $\Omega \subset \mathbb{R}^d$, with

- $\phi : \Omega \to \mathbb{R}^d$ elastic deformation field
- $z : \Omega \to \mathbb{R}^m$ internal variable: a *mesoscopic averaged phase variable*, e.g. ....

The energy functional $I(t, \phi, z) = \mathcal{E}_1(z) + I_2(t, \phi, z)$

$$\mathcal{E}_1(z) = \int_\Omega \frac{1}{q} |\nabla z|^q + I_{\mathrm{K}}(z) \quad q > d, \mathrm{K} \text{ convex set},$$

$$I_2(t, \phi, z) = \int_\Omega W(\nabla \phi(x), z(x)) - \langle \ell(t), \phi \rangle_{W^{1,p}}$$

$W(\cdot, z)$ polycvx, p-coerc., ℓ smooth ext. loading

The dissipation potential

$$\Psi(v) = \int_\Omega \psi(v(x))\,\mathrm{d}x \qquad \psi : \mathbb{R} \to \mathbb{R} \text{ cvx, with superlinear growth}$$

The PDE system

$$\Psi(v) = \int_\Omega \psi(v(x))\,\mathrm{d}x$$

$$I(t,\phi,z) = \mathcal{E}_1(z) + I_2(t,\phi,z) \quad \text{with} \quad \begin{array}{l} \mathcal{E}_1(z) = \int_\Omega \frac{1}{q}|\nabla z|^q + I_{\mathrm{K}}(z) \\ I_2(t,\phi,z) = \int_\Omega W(\nabla\phi(x), z(x)) - \langle \ell(t), \phi\rangle_{W^{1,p}} \end{array}$$

♣ Consider the PDE system with unknowns (z,ϕ)

$$\partial\psi(\dot z) - \Delta_q z + \partial I_{\mathrm{K}}(z) + \mathrm{D}_z W(\nabla\phi, z) \ni 0 \quad \text{a.e. in } (0,T)\times\Omega,$$

$$I(t,\phi,z) \le I(t,\eta,z) \quad \text{for all } \eta \in \underbrace{\{\phi \in W^{1,p}(\Omega;\mathbb{R}^d) \,:\, \phi = \phi_{\mathrm{Dir}} \text{ on } \Gamma_{\mathrm{Dir}}\}}_{\doteq\, \mathcal{F}}$$

The PDE system

♣ Consider the PDE system with unknowns (z, ϕ)

$$\partial\psi(\dot{z}) - \Delta_q z + \partial I_{\mathrm{K}}(z) + \mathrm{D}_z W(\nabla\phi, z) \ni 0 \quad \text{a.e. in } (0,T) \times \Omega,$$

$$I(t,\phi,z) \leq I(t,\eta,z) \quad \text{for all } \eta \in \{\phi \in W^{1,p}(\Omega;\mathbb{R}^d) \, : \, \phi = \phi_{\mathrm{Dir}} \text{ on } \Gamma_{\mathrm{Dir}}\} \doteq \mathcal{F}$$

An abstract approach

Consider the **marginal** energy

$$\mathcal{E}(t,z) := \min_{\phi\in\mathcal{F}} I(t,\phi,z) = \mathcal{E}_1(z) + \min_{\phi\in\mathcal{F}} I_2(t,\phi,z) \doteq \mathcal{E}_1(z) + \mathcal{E}_2(t,z)$$

and the doubly nonlinear equation (in $B = L^2(\Omega;\mathbb{R}^m)$)

$$\partial\Psi(z'(t)) + \partial_z\mathcal{E}(t,z(t)) \ni 0 \quad \text{a.e. in } (0,T)$$

The PDE system

♣ Consider the PDE system with unknowns (z, ϕ)

$$\partial\psi(\dot{z}) - \Delta_q z + \partial I_{\mathrm{K}}(z) + \mathrm{D}_z W(\nabla\phi, z) \ni 0 \quad \text{a.e. in } (0,T)\times\Omega,$$

$$I(t,\phi,z) \leq I(t,\eta,z) \quad \text{for all } \eta \in \{\phi \in W^{1,p}(\Omega;\mathbb{R}^d) \,:\, \phi = \phi_{\mathrm{Dir}} \text{ on } \Gamma_{\mathrm{Dir}}\} \doteq \mathcal{F}$$

An abstract approach

Consider the **marginal** energy

$$\mathcal{E}(t,z) := \min_{\phi\in\mathcal{F}} I(t,\phi,z) = \mathcal{E}_1(z) + \min_{\phi\in\mathcal{F}} I_2(t,\phi,z) \doteq \mathcal{E}_1(z) + \mathcal{E}_2(t,z)$$

and the doubly nonlinear equation (in $B = L^2(\Omega;\mathbb{R}^m)$)

$$\begin{aligned} &\partial\Psi(z'(t)) + \partial_z\mathcal{E}(t,z(t)) \ni 0 \quad \text{a.e. in } (0,T) \\ &\quad \text{with } \partial_z\mathcal{E}(t,z) = \boldsymbol{\partial_z}^{\mathbf{co}}\mathcal{E}_1(z) + \boldsymbol{\partial_z}^{\mathbf{ma}}\mathcal{E}_2(t,z) \\ &\quad \boldsymbol{\partial_z}^{\mathbf{ma}}\mathcal{E}_2(t,z) = \{\mathrm{D}_z I_2(t,\phi,z) \,:\, \phi \in \mathrm{Argmin} I_2(t,\cdot,z)\} \end{aligned} \tag{DNE}$$

Solutions to (DNE) solve the PDE system!

An abstract approach

$$\partial\Psi(z'(t)) + \partial_z \mathcal{E}(t, z(t)) \ni 0 \quad \text{a.e. in } (0, T)$$

$$\partial_z \mathcal{E}(t, z) = \boldsymbol{\partial_z}^{\mathbf{co}} \mathcal{E}_1(z) + \boldsymbol{\partial_z}^{\mathbf{ma}} \mathcal{E}_2(t, z) \qquad \text{(DNE)}$$

$$\boldsymbol{\partial_z}^{\mathbf{ma}} \mathcal{E}_2(t, z) = \{\mathrm{D}_z I_2(t, \phi, z) \ : \ \phi \in \mathrm{Argmin} I_2(t, \cdot, z)\}$$

♣ The general existence theorem applies to (DNE):

- $\partial_z \mathcal{E}$ complies with
  Euler equation for minimimum problems
  strong-weak closedness
  chain rule inequality
- use as surrogate for time derivative $\partial_t \mathcal{E}$:

$$\mathrm{P}_t(t, z, \xi) := \sup \{\partial_t I_2(t, \phi, z) \ : \ \phi \in \mathrm{Argmin} I_2(t, \cdot, z), \ \xi = \mathrm{D}_z I_2(t, \phi, z)\}$$

$$\text{for } \xi \in \boldsymbol{\partial_z}^{\mathbf{ma}} \mathcal{E}_2(t, z).$$