Existence and uniqueness results for a class of rate-independent hysteresis problems

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Problem:

Existence and **uniqueness** for the Cauchy problem for (DNE).

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Note:

Ψ also depends on the state variable z!!!

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Note:

Ψ has linear growth at ∞!!!

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Note:

The problem is **rate-independent!!!**

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Since Ψ is **1-homogeneous,**

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Hysteresis effects may appear.

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In rate-independent models,

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Our strategy:

Exploit the rate-independent character of the problem both for existence and uniqueness!!

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To simplify:

We consider the case of **state-independent** Ψ and **smooth** $\mathcal{E}$, $\mathcal{E} \in C^1([0,T] \times Z; \mathbb{R}^+)$.

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$$\partial\Psi(\dot{z}(t)) + D\mathcal{E}(t, z(t)) \ni 0, \quad t \in (0, T),$$

In this case:

- existence
- approximation via variable time-step discretization
- uniqueness & continuous dependence on the initial data
- strong convergence and error estimates for the approximate solutions

in [**Mielke-Theil**, *On rate-independent models*, NoDEA 2004].

The Energetic Formulation

$$\begin{cases} \partial\Psi(\dot{z}(t)) + D\mathcal{E}(t, z(t)) \ni 0, \quad t \in (0, T), \\ z(0) = z_0 \end{cases} \tag{SF}$$

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We will prove:

Any solution $z \in W^{1,1}(0, T; Z)$ of (SF) fulfils

the **stability condition**

$$\mathcal{E}(t, z(t)) \leq \mathcal{E}(t, y) + \Psi(y - z(t)) \quad \forall y \in Z, \quad \text{for a.e. } t \in (0, T)$$

and for all $t \in [0, T]$ the **energy balance**

$$\int_0^t \Psi(\dot{z}(r))dr + \mathcal{E}(t, z(t)) = \mathcal{E}(0, z_0) + \int_0^t \partial_r \mathcal{E}(r, z(r))dr.$$

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Crucial facts on Ψ:

By convexity and 1-positive homogeneity, $\exists$ a closed convex subset $\mathcal{C} \subset Z'$ s.t.

$$\begin{cases} \Psi(v) = \max\{\langle\sigma, v\rangle \mid \sigma \in \mathcal{C}\} \,\forall v \in Z \\ \text{i.e., } \Psi \text{ is the } \textbf{support function} \text{ of } \mathcal{C}, \\ \partial\Psi(v) = \operatorname{argmax}\{\langle\sigma, v\rangle \mid \sigma \in \mathcal{C}\} = (\partial I_{\mathcal{C}})^{-1}(v), \\ \partial\Psi(v) \subset \mathcal{C} = \partial\Psi(0). \end{cases}$$

Towards the energetic formulation: **stability**

- Then, we reformulate the doubly nonlinear equation as

$$-D\mathcal{E}(t, z(t)) \subset \partial\Psi(\dot{z}(t)) \subset \partial\Psi(0) \quad t \in (0, T). \qquad \text{(DNE2)}$$

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- Fix $y \in Z$ and test (DNE2) by $y - z(t)$: since $\Psi(0) = 0$,

$$\begin{aligned}\Psi(y - z(t)) = \Psi(y - z(t)) - \Psi(0) &\geq -\langle D\mathcal{E}(t, z(t)), y - z(t)\rangle \\ &\geq \mathcal{E}(t, z(t)) - \mathcal{E}(t, y).\end{aligned}$$

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- Hence, we obtain the **stability condition**

$$\mathcal{E}(t, z(t)) \leq \mathcal{E}(t, y) + \Psi(y - z(t)) \quad \forall y \in Z \quad \text{for a.e. } t \in (0, T)$$

Towards the energetic formulation: **energy balance**

- We test

$$-D\mathcal{E}(t, z(t)) \subset \partial\Psi(0) \quad \text{for a.e. } t \in (0, T)$$

by $\dot{z}(t)$, whence

$$\Psi(\dot{z}(t)) + \langle D\mathcal{E}(t, z(t)), \dot{z}(t)\rangle \geq 0.$$

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Towards the energetic formulation: **energy balance**

- We conclude the **differential form** of the **energy balance:**

$$\Psi(\dot{z}(t)) + \langle D\mathcal{E}(t, z(t)), \dot{z}(t)\rangle = 0 \quad \text{for a.e. } t \in (0, T), \qquad \text{(Eloc)}$$

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- Recall the **chain rule** formula

$$\frac{d}{dt}\mathcal{E}(t, z(t)) = \langle D\mathcal{E}(t, z(t)), \dot{z}(t)\rangle + \partial_t \mathcal{E}(t, z(t)),$$

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- Integrate (Eloc) on the time interval (s, t), $0 \leq s \leq t \leq T$.

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- We obtain the **energy balance** for all $0 \leq s \leq t \leq T$:

$$\int_s^t \Psi(\dot{z}(r))dr + \mathcal{E}(t, z(t)) = \mathcal{E}(s, z(s)) + \int_s^t \partial_r \mathcal{E}(r, z(r))dr.$$

The Energetic Formulation

Crucial Fact:

If $z \in W^{1,1}(0, T; Z)$, the **stability condition** and the **energy balance** are **equivalent** to the pointwise, subdifferential formulation

$$\partial\Psi(\dot{z}(t)) + D\mathcal{E}(t, z(t)) \ni 0 \quad t \in (0, T).$$

Back to our problem

$$\begin{cases} \partial\Psi(z(t),\dot{z}(t)) + \partial\mathcal{E}(t,z(t)) \ni 0 \quad \text{for a.e. } t \in (0,T), \\ z(0) = z_0 \end{cases} \tag{SF}$$

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Stability for a.e. $t \in (0, T)$

$$\mathcal{E}(t, z(t)) \leq \mathcal{E}(t, y) + \Psi(z(t), y - z(t)) \quad \forall y \in Z,$$

and the energy balance for all $t \in [0, T]$

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Equivalence of Subdifferential and Energetic formulations

$z \in W^{1,1}(0, T; Z)$ is a solution to (SF) iff it fulfils the stability condition for a.e. $t \in (0, T)$

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Same proof: use a **chain rule for $\partial\mathcal{E}$!**

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[**Mielke-R.**, 2005, to appear on *M3AS*]:

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$$\frac{\alpha}{2}\theta(1-\theta)\|z_0 - z_1\|^2 \leq (1-\theta)\mathcal{E}(t, z_0) + \theta\mathcal{E}(t, z_1) \\ -\mathcal{E}(t, (1-\theta)z_0 + \theta z_1) \quad \forall 0 \leq \theta \leq 1,$$

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- $\exists C_\Psi > 0$ s.t. $\Psi(z, v) \leq C_\Psi \|v\|$ for all $(z, v) \in Z \times Z$;

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- $\exists C_\Psi > 0$ s.t. $\Psi(z, v) \le C_\Psi \|v\|$ for all $(z, v) \in Z \times Z$;
- $\exists 0 < \psi^* < \alpha : \quad |\Psi(z, v) - \Psi(\hat{z}, v)| \le \psi^* \|v\| \quad \forall z, \hat{z}, v \in Z,$

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Theorem

- $\mathcal{E} : (0, T) \times Z \to \mathbb{R}$ bounded from below and l.s.c.;
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- $$\exists 0 < \psi^* < \alpha : \quad |\Psi(z, v) - \Psi(\hat{z}, v)| \leq \psi^* \|v\| \quad \forall z, \hat{z}, v \in Z,$$

Then, for any **stable** initial datum z_0, **the energetic formulation admits a solution** $z \in W^{1,\infty}(0, T; Z)$.

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Approximation and passage to the limit

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- passage to the limit via **weak compactness, lower semicontinuity, Young measures arguments** for a **uniform time-step** approximation;
- stability condition and energy identity in the limit (via the **chain rule**) $\Rightarrow$ existence of a solution to the energetic formulation

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are **tightly related**

Mielke&Theil's energetic estimate (in the state-independent case!)

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Continuous dependence argument of **[Mielke-Theil, 2004]**: let z_1 and z_2 be two $W^{1,1}(0,T;Z)$ solutions of

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Since $\mathcal{E}$ is **smooth** and α-**uniformly convex**,

$$\alpha\|z_1(t)-z_2(t)\|^2 \leq \gamma(t) := \langle D\mathcal{E}(t,z_1(t))-D\mathcal{E}(t,z_2(t)), z_1(t)-z_2(t)\rangle.$$

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[Mielke-Theil'04]: By the **smoothness** of $\mathcal{E}$ and the **1-homogeneity** of $\Psi(z,\cdot)$, obtain the Gronwall estimate

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and conclude **Lipschitz continuous dependence** on the data!

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Hence,

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The proof of **uniqueness is highly nontrivial**!

The BROKATE-KREJČÍ-SCHNABEL argument

First uniqueness and continuous dependence result for **quasi-variational** sweeping processes $\rightsquigarrow$ [**Brokate-Krejčí-Schnabel**, J. Convex Anal. 2004]:

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Assume Z Hilbert and $\mathcal{E}(t,z) :\sim \|z\|^2$

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Main ideas:

- Exploit convex analysis results and the **support function** properties of $\Psi(z,\cdot)$ to deduce a **more suitable representation** of $\partial I_{\mathcal{C}(z(t))}$;

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How to deal with a general $\mathcal{E}$??

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- **But** our **continuous dependence** result is **intermediate**: does not entirely cover the Brokate-Krejcí-Schnabel's and the Mielke-Theil's results.

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