Thermal effects in adhesive contact: modelling and analysis

Elena Bonetti*, Giovanna Bonfanti†, Riccarda Rossi‡

Abstract

In this paper, we consider a contact problem with adhesion between a viscoelastic body and a rigid support, taking thermal effects into account. The PDE system we deal with is derived within the modelling approach proposed by M. Frémond and, in particular, includes the entropy balance equations, describing the evolution of the temperatures of the body and of the adhesive material. Our main result consists in showing the existence of global in time solutions (to a suitable variational formulation) of the related initial and boundary value problem.

Key words: contact, adhesion, entropy balance, thermoviscoelasticity, global in time existence of solutions.

AMS (MOS) Subject Classification: 35K55, 35Q72, 74A15, 74M15.

1 Introduction

This paper addresses the analysis of adhesive contact between a viscoelastic body and a rigid support, in the case when thermal effects are included. Contact with adhesion is described using the modelling approach proposed by Frémond (see [21, Chap. 14]), which was originally introduced for the isothermal case, combining the theory of damage (see, e.g., [22], [12], [21, Chap. 12]) with the theory of unilateral contact. Indeed, although the unilateral contact theory (which prescribes the impenetrability condition between the bodies) does not allow for any resistance to tension, in the adhesion phenomenon resistance to tension is given by micro-bonds on the contact surface, preventing separation. Adhesion is active if these bonds (one may think of a "glue" on the contact surface) are not damaged. Thus, the description of this phenomenon has to take into account the state of the adhesive bonds (through a "damage parameter") and the microscopic movements breaking them, as well as macroscopic deformations and displacements.

In the recent papers [4] and [5], we have introduced the model and derived the corresponding initial and boundary value problem in the isothermal case. The resulting PDE system couples an equation for macroscopic deformations of the body and a "boundary" equation on the contact surface, describing the evolution of the state of the glue by a surface damage parameter. The system is highly nonlinear, mainly due to the presence of nonlinear boundary conditions and nonsmooth constraints on the physical variables. In [4], existence of a global in time solution for a weak version of the corresponding PDE system was proved in the case of irreversible damage

*Dipartimento di Matematica "F. Casorati", Università di Pavia. Via Ferrata 1, I–27100 Pavia, Italy. Email: elena.bonetti@unipv.it

†Dipartimento di Matematica, Università di Brescia. Via Valotti 9, I–25133 Brescia, Italy. E-mail: bonfanti@ing.unibs.it

‡Dipartimento di Matematica, Università di Brescia. Via Valotti 9, I–25133 Brescia, Italy. E-mail: riccarda.rossi@ing.unibs.it

dynamics for the glue. In the subsequent contribution [5], focusing on the reversible case, we proved well-posedness results and further investigated the long-time behaviour of the solutions.

In this paper, we aim to generalize the model introduced in [4] and [5], including thermal effects both on the contact surface, and in the interior. We believe this to be interesting from the modeling perspective, because external thermal actions can in fact influence the state of the adhesive material, see [21].

In extending the model to the non-isothermal case, we shall adopt the following viewpoint: we shall assume that the body temperature and the glue temperature may be different and thus governed by two distinct entropy balance laws.

1.1 The model and the PDE system

Let us now introduce the model and derive the corresponding initial and boundary value problem. On a time interval $(0,T)$, we investigate the mechanical evolution of a thermoviscoelastic body located in a smooth bounded domain $\Omega \subset \mathbb{R}^3$, whose boundary is $\partial\Omega = \overline{\Gamma}_1 \cup \overline{\Gamma}_2 \cup \overline{\Gamma}_c$. Here Γ_i, $i = 1, 2, c$, are open subsets in the relative topology of $\partial\Omega$, each of them with a smooth boundary and disjoint one from each other. In particular, Γ_c is the contact surface. Hereafter we shall suppose that both Γ_c and Γ_1 have positive measure. Without loss of generality , we shall treat Γ_c as a flat surface and identify it with a subset of $\mathbb{R}^2$.

The thermomechanical equilibrium of the system is described by the state variables. We consider the absolute temperature of the body ϑ and the symmetric linearized strain tensor $\epsilon(\mathbf{u})$ (we shall denote by $\mathbf{u}$ the vector of small displacements), defined in $\Omega \times (0,T)$. Moreover, the variables describing the equilibrium on $\Gamma_c \times (0,T)$ are the absolute temperature of the glue ϑ_s, a damage parameter χ, its gradient $\nabla\chi$, and the trace $\mathbf{u}_{|\Gamma_c}$ of the displacement $\mathbf{u}$ on the contact surface. The parameter χ is assumed to take values in $[0,1]$, with $\chi = 0$ for completely damaged bonds, $\chi = 1$ for undamaged bonds, and $\chi \in (0,1)$ for partially damaged bonds.

The free energy of the system is given by a volume contribution Ψ_Ω and a surface one Ψ_{Γ_c}. It is known from thermodynamics that the free energy is concave with respect to the temperature. Thus, considering a fairly general expression for the purely thermal contribution in the free energy (cf. [11]) and normalizing some physical constants, we assume in $\Omega \times (0,T)$

$$\Psi_\Omega = -j(\vartheta) + p(\vartheta)\mathrm{tr}(\epsilon(\mathbf{u})) + \frac{1}{2}\epsilon(\mathbf{u})K\epsilon(\mathbf{u}), \tag{1.1}$$

where j is a sufficiently regular, increasing, and convex real function, the function p accounts for the thermal expansion energy, and $K = (a_{ijkh})$ denotes the elasticity tensor for a possibly anisotropic and inhomogeneous material. In the sequel, for the sake of simplicity, we assume $p(\vartheta) = \vartheta$. A few comments on the function j are now in order. A possible choice for j, often used in the literature, is

$$j(\vartheta) = \vartheta \log\vartheta - \vartheta. \tag{1.2}$$

This enforces the physical constraint that ϑ be strictly positive. However, in our mathematical analysis we are going to tackle more general situations. In particular, we shall not require any condition on the domain of j. Analogously, we prescribe in $\Gamma_c \times (0,T)$

$$\Psi_{\Gamma_c} = -j(\vartheta_s) + \lambda(\chi)(\vartheta_s - \vartheta_{eq}) + I_{[0,1]}(\chi) + \sigma(\chi) + \frac{1}{2}|\nabla\chi|^2 + \frac{1}{2}\chi|\mathbf{u}_{|\Gamma_c}|^2 + I_-(\mathbf{u}_{|\Gamma_c}\cdot\mathbf{n}), \tag{1.3}$$

where $\vartheta_{eq} > 0$ is a critical temperature and λ is a regular (quadratic) function. Once we consider contact with adhesion as the effect of a phase transition between the undamaged and damaged state of the adhesive substance on the contact surface, λ' formally corresponds to the so-called latent heat in phase transitions models and ϑ_{eq} to the critical temperature between undamaged and damaged adhesion. This relates to the assumption that just by temperature devices we can damage the micro-bonds on the contact surface. Moreover, the indicator function $I_{[0,1]}$ of the interval $[0,1]$ accounts for physical constraints on χ, being $I_{[0,1]}(\chi) = 0$ if $\chi \in [0,1]$ and $I_{[0,1]}(\chi) = +\infty$ otherwise. Analogously, denoting by I_- the indicator function of the interval $(-\infty, 0]$, the term $I_-(\mathbf{u}_{|\Gamma_c} \cdot \mathbf{n})$ renders the impenetrability condition on the contact surface, as it enforces that $\mathbf{u}_{|\Gamma_c} \cdot \mathbf{n} \leq 0$ ($\mathbf{n}$ is the outward unit normal vector to Γ_c). Finally, the function σ is sufficiently smooth and possibly nonconvex, being related to nonmonotone dynamics for χ (from a physical point of view, it corresponds to some cohesion in the material).

The free energy describes the thermomechanical equilibrium of the system in terms of fixed state variables. Hence, we follow the approach proposed by J.J. Moreau to prescribe the dissipated energy by means of a dissipation functional, the so-called pseudo-potential of dissipation, which is a convex, nonnegative functional, attaining its minimum 0 when the dissipation (described by the dissipative variables) is zero. The dissipative variables defined in $\Omega \times (0,T)$ are $\nabla\vartheta$ and $\epsilon(\mathbf{u}_t)$. Thus, we define the volume part Φ_Ω of the pseudo-potential of dissipation by

$$\Phi_\Omega = \frac{1}{2}|\nabla\vartheta|^2 + \frac{1}{2}\epsilon(\mathbf{u}_t)K_v\epsilon(\mathbf{u}_t), \tag{1.4}$$

where $K_v = (b_{ijkh})$ denotes the viscosity tensor for a possibly anisotropic and inhomogeneous material. The surface part Φ_{Γ_c} of the pseudo-potential of dissipation depends on $\nabla\vartheta_s$, χ_t, and also on the difference $(\vartheta_{|\Gamma_c} - \vartheta_s)$ between the temperatures of the body and of the glue on the contact surface, namely

$$\Phi_{\Gamma_c} = \frac{1}{2}|\nabla\vartheta_s|^2 + \frac{1}{2}|\chi_t|^2 + \frac{1}{2}k(\chi)(\vartheta_{|\Gamma_c} - \vartheta_s)^2. \tag{1.5}$$

Here, k is a sufficiently regular function and its physical meaning is related to the heat exchange between the body and the adhesive material. It is of fairly natural evidence (see also [15, 30, 31] that the possibility (and the amount) of heat exchange between the body and the contact surface depends on the fact that the adhesion is more or less active. We let k to be nonnegative (in accordance with thermodynamical consistency ensured by the convexity of the pseudo-potential of dissipation), increasing, and possibly vanishing when χ attains its minimum value 0. Indeed, we may think that if the adhesion is not active no heat exchange is allowed ($k(0) = 0$) or that a residual heat exchange is preserved even for the completely damaged adhesive substance ($k(0) > 0$).

Now, let us introduce the equations in accordance with the laws of thermomechanics. We consider the momentum balance (in the quasi-static case)

$$-\operatorname{div}\Sigma = \mathbf{f} \quad \text{in } \Omega \times (0,T), \tag{1.6}$$

where Σ is the stress tensor, combined with the boundary conditions ($\mathbf{R}$ is the reaction on the contact surface)

$$\Sigma\mathbf{n} = \mathbf{R} \quad \text{in } \Gamma_c \times (0,T), \qquad \mathbf{u} = \mathbf{0} \quad \text{in } \Gamma_1 \times (0,T), \qquad \Sigma\mathbf{n} = \mathbf{g} \quad \text{in } \Gamma_2 \times (0,T), \tag{1.7}$$

$\mathbf{f}$ being a volume force and $\mathbf{g}$ a traction. The thermal balance is given by the following entropy equation

$$s_t + \operatorname{div}\mathbf{Q} = h \quad \text{in } \Omega \times (0,T), \tag{1.8}$$

s denoting the internal entropy, $\mathbf{Q}$ the entropy flux, and h and external entropy source. Indeed, equation (1.8) can be obtained rescaling the first law of thermodynamics (dividing the internal energy balance by the absolute temperature), under the small perturbation assumption. We refer to [10], [11] and [7], [8], [9] for details on this modelling approach and related analytical results. Moreover, we supplement (1.8) with the following boundary conditions

$$\mathbf{Q}\cdot\mathbf{n} = F \text{ on } \Gamma_c \times (0,T), \quad \mathbf{Q}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega \setminus \Gamma_c \times (0,T), \tag{1.9}$$

where F denotes the entropy flux through Γ_c. Hence, on the contact surface, we introduce a balance equation for the microscopic movements, that is

$$B - \operatorname{div}\mathbf{H} = 0 \text{ in } \Gamma_c \times (0,T), \quad \mathbf{H}\cdot\mathbf{n}_s = 0 \text{ on } \partial\Gamma_c \times (0,T), \tag{1.10}$$

B, $\mathbf{H}$ representing interior forces, responsible for the damage of adhesive bonds between the body and the support, and $\mathbf{n}_s$ the outward unit normal vector to $\partial\Gamma_c$. Then, the entropy equation on the contact surface is given by

$$\partial_t s_s + \operatorname{div}\mathbf{Q}_s = F \text{ in } \Gamma_c \times (0,T), \quad \mathbf{Q}_s\cdot\mathbf{n}_s = 0 \text{ on } \partial\Gamma_c \times (0,T). \tag{1.11}$$

Here, s_s is the entropy on the contact surface, $\mathbf{Q}_s$ the surface entropy flux, and the term F, given by the flux through the boundary Γ_c (cf. (1.9)), represents a surface entropy source.

Constitutive relations for $\Sigma, \mathbf{R}, s, \mathbf{Q}, F, B, \mathbf{H}, s_s, \mathbf{Q}_s$ are given in terms of the free energies and the pseudo-potentials of dissipation. More precisely, we have

$$s = -\frac{\partial\Psi_\Omega}{\partial\vartheta} = \ell(\vartheta) - \operatorname{div}\mathbf{u}, \tag{1.12}$$

$$s_s = -\frac{\partial\Psi_{\Gamma_c}}{\partial\vartheta_s} = \ell(\vartheta_s) - \lambda(\chi), \tag{1.13}$$

where ℓ is the derivative of the convex function j. In the physical case $j(x) = x\log x - x$ (cf. (1.2)), we have

$$\ell(x) = \log x. \tag{1.14}$$

In particular, ℓ in (1.14) in fact yields an internal positivity constraint on the system temperatures ϑ and ϑ_s. Furthermore,

$$\mathbf{Q} = -\frac{\partial\Phi_\Omega}{\partial\nabla\vartheta} = -\nabla\vartheta, \tag{1.15}$$

$$\mathbf{Q}_s = -\frac{\partial\Phi_{\Gamma_c}}{\partial\nabla\vartheta_s} = -\nabla\vartheta_s, \tag{1.16}$$

$$F = \frac{\partial\Phi_{\Gamma_c}}{\partial(\vartheta_{|\Gamma_c} - \vartheta_s)} = k(\chi)(\vartheta_{|\Gamma_c} - \vartheta_s). \tag{1.17}$$

The constitutive relation for the stress tensor Σ accounts for dissipative (viscous) dynamics for deformations, in that we have

$$\Sigma = \frac{\partial\Psi_\Omega}{\partial\epsilon(\mathbf{u})} + \frac{\partial\Phi_\Omega}{\partial\epsilon(\mathbf{u}_t)} = K\epsilon(\mathbf{u}) + K_v\epsilon(\mathbf{u}_t) + \vartheta\mathbf{1}, \tag{1.18}$$

($\mathbf{1}$ denotes the identity matrix), while the reaction $\mathbf{R}$ is given by

$$\mathbf{R} = -\frac{\partial\Psi_{\Gamma_c}}{\partial\mathbf{u}_{|\Gamma_c}} = -\chi\mathbf{u}_{|\Gamma_c} - \partial I_-(\mathbf{u}_{|\Gamma_c}\cdot\mathbf{n})\mathbf{n}. \tag{1.19}$$

We further prescribe B

$$
\begin{aligned}
B &= \frac{\partial \Psi_{\Gamma_c}}{\partial \chi} + \frac{\partial \Phi_{\Gamma_c}}{\partial \chi_t} \\
&= \lambda'(\chi)(\vartheta_s - \vartheta_{eq}) + \partial I_{[0,1]}(\chi) + \sigma'(\chi) + \frac{1}{2}|\mathbf{u}_{|\Gamma_c}|^2 + \chi_t,
\end{aligned}
\tag{1.20}
$$

(∂I_- and $\partial I_{[0,1]}$ standing for the subdifferentials of I_- and $I_{[0,1]}$, respectively), and let $\mathbf{H}$ be

$$
\mathbf{H} = \frac{\partial \Psi_{\Gamma_c}}{\partial \nabla \chi} = \nabla \chi. \tag{1.21}
$$

Remark 1.1. Let us point out that the evolution of the system is characterized by dissipation due to choice of dissipative potentials Φ_Ω and Φ_{Γ_c} (cf. (1.4) and (1.5)) and the balance laws of thermodynamics (see in particular (1.8) and (1.11), (1.6), and (1.10)).

Combining the previous constitutive relations with the balance laws, we obtain the following boundary value problem

$$
\partial_t(\ell(\vartheta)) - \operatorname{div}(\mathbf{u}_t) - \Delta\vartheta = h \qquad \text{in } \Omega \times (0,T), \tag{1.22}
$$

$$
\partial_n \vartheta = \begin{cases} 0 & \text{in } (\partial\Omega \setminus \Gamma_c) \times (0,T), \\ -k(\chi)(\vartheta - \vartheta_s) & \text{in } \Gamma_c \times (0,T), \end{cases} \tag{1.23}
$$

$$
\partial_t(\ell(\vartheta_s)) - \partial_t(\lambda(\chi)) - \Delta\vartheta_s = k(\chi)(\vartheta - \vartheta_s) \qquad \text{in } \Gamma_c \times (0,T), \tag{1.24}
$$

$$
\partial_n \vartheta_s = 0 \qquad \text{in } \partial\Gamma_c \times (0,T), \tag{1.25}
$$

$$
-\operatorname{div}\,(K\varepsilon(\mathbf{u}) + K_v\varepsilon(\mathbf{u}_t) + \vartheta\mathbf{1}) = \mathbf{f} \qquad \text{in } \Omega \times (0,T), \tag{1.26}
$$

$$
\mathbf{u} = \mathbf{0} \quad \text{in } \Gamma_1 \times (0,T), \quad (K\varepsilon(\mathbf{u}) + K_v\varepsilon(\mathbf{u}_t) + \vartheta\mathbf{1})\mathbf{n} = \mathbf{g} \qquad \text{in } \Gamma_2 \times (0,T), \tag{1.27}
$$

$$
(K\varepsilon(\mathbf{u}) + K_v\varepsilon(\mathbf{u}_t) + \vartheta\mathbf{1})\mathbf{n} + \chi\mathbf{u} + \partial I_-(\mathbf{u}\cdot\mathbf{n})\mathbf{n} \ni \mathbf{0} \qquad \text{in } \Gamma_c \times (0,T), \tag{1.28}
$$

$$
\chi_t - \Delta\chi + \partial I_{[0,1]}(\chi) + \sigma'(\chi) - \lambda'(\chi)\vartheta_{\mathrm{eq}} \ni -\lambda'(\chi)\vartheta_s - \frac{1}{2}|\mathbf{u}|^2 \qquad \text{in } \Gamma_c \times (0,T), \tag{1.29}
$$

$$
\partial_{\mathbf{n}}\chi = 0 \qquad \text{in } \partial\Gamma_c \times (0,T) \tag{1.30}
$$

(here and in what follows, we shall omit for simplicity the index $v_{|\Gamma_c}$ to denote the trace on Γ_c of a function v, defined in Ω).

Remark 1.2. As we have already pointed out, our analysis actually accounts for a form of the thermal contribution in the free energies more general than (1.2) (cf. (1.1) and (1.3)). Thus, our results may apply to several physical situations with different thermal behaviour. We note that we can also handle the case when the specific heat c_V, given by the thermodynamic relation $c_V = -\vartheta \frac{\partial^2 \Psi_\Omega}{\partial \vartheta^2}$ (cf. [11, Rem. 2.1]), is not constant. Indeed, it is known from physics that the specific heat may be depending on the temperature, e.g., $c_V(\vartheta) = \vartheta^\gamma$, $\gamma > 0$. A choice of j corresponding to the latter expression of c_V is

$$
j(\vartheta) = \frac{\vartheta^{\gamma+1}}{\gamma(\gamma+1)},
$$

which is covered by our analysis. In particular, letting $\gamma = 1$ (and hence $j(\vartheta) = \frac{1}{2}\vartheta^2$), we get $\ell(\vartheta) = \vartheta$, so that (1.22) and (1.24) reduce to Caginalp-type heat equations (cf. with [16]).

1.2 Related literature and our own results

For a review of the theory of contact problems see, e.g., the monographs [27, 19, 29], and the references therein. We also refer to [4, 5] for some partial survey of the literature on (isothermal) models of adhesive contact. In this connection, we mention [24], where a unilateral contact model (also derived within Frémond's approach) is considered. Therein, the adhesive properties are described by the condition $\chi\mathbf{u} = 0$ on the contact surface. The author proves an existence theorem for the related PDE system and also develops some numerical investigations. The paper [25] focuses on a model combining unilateral contact with adhesion and friction as well: under a smallness condition on the friction coefficient, an existence result is proved and various numerical schemes are proposed.

As for the literature on contact models including thermal effects in the three-dimensional framework, besides the contributions mentioned in [19] we may recall [26], where a frictionless contact problem between a thermoelastic body and a rigid foundation is modelled by a parabolic equation for ϑ, coupled with an elliptic equation for $\mathbf{u}$, with mixed boundary conditions. An existence result is proved, provided the coefficient of thermal expansion is sufficiently small. Among dynamic models, in which contact is rendered by means of a normal compliance condition, we quote [20], which deals with a wide class of frictional contact problems in thermoelasticity and thermoviscoelasticity. Moreover, we mention [1], where a frictional contact problem involving a thermoelastic body undergoing wear on the contact surface is investigated. A well-posedness result is proved for a system coupling a parabolic equation for the temperature, a variational inequality for the displacement, and a first order equation for the wear function, supplemented with nonlinear boundary conditions.

Nevertheless, as far as we know, no results are available, in the literature, on unilateral contact models which take into account both adhesive properties and thermal effects. Indeed, one of the main novelties of the present contribution is that we consider heat generation effects in the adhesive contact phenomenon, too. That is to say, we allow for the body and the adhesive material to have different temperatures, whose evolution is mainly ruled by the heat exchange throughout the contact surface. More precisely, the entropy flux F through Γ_c (occurring in (1.9)) plays the role of a source of entropy in (1.11). From an analytical point of view, this results in a nonlinear coupling between (1.22)–(1.23) and (1.24) and gives raise to some technical difficulties. A further peculiarity of our work consists in assuming entropy balance laws (in place of the more usual internal energy balance), for describing the evolution of the body and of the glue temperatures. This brings to strong and possibly singular nonlinearities in (1.22) and (1.24) (see (1.14)). An advantage of this choice is that, assuming that the domain $D(j) \subseteq (0, +\infty)$ (as in the case of the classical choice (1.2)) once the problem is solved in a suitable sense, the positivity of the temperatures is deduced. On this fact the thermodynamical consistency of the model relies, see also [10, 7, 8, 11]. This is of particular interest in the present case, since the low spatial regularity of the solution components ϑ and ϑ_s, along with the nonlinear boundary condition (1.23), prevents us from using any maximum principle.

In fact, we shall study the Cauchy problem for a *generalized version* of system (1.22)–(1.30), see Problem (**P**) in Sec. 2.3. Namely, we replace the subdifferential operators in (1.28)–(1.29), by general maximal monotone operators (possibly rendering physical constraints on the variables χ and $\mathbf{u}$). Further, we generalize the choice of the nonlinearity ℓ, allowing for a maximal monotone operator. In fact, the only restriction we impose on ℓ is that the resulting internal energy of the system be coercive, cf. with (1.31) below. This is reasonable from a physical point of view and still enables us to include several choices of ℓ in our analysis (in particular, (1.2), as well as the examples of Remark 1.2). The idea is that, once the internal energy of the system is bounded, the absolute temperature is bounded, too. Let us focus on the volume temperature ϑ. As known from thermodynamics, the internal energy (depending on the entropy s) can be introduced as the convex conjugate function (with respect to the variable ϑ), of the negative of the free energy (which is convex w.r.t. the temperature as the function j is convex, see (1.1) and (1.3)). Namely, in the case of the volume free energy the related internal energy is

$$e(s,\cdot) = (-\Psi_\Omega)^*(s,\cdot) = \sup_\vartheta (s\vartheta + \Psi_\Omega(\vartheta,\cdot)).$$

Thus, our coercivity condition may be expressed in terms of the conjugate of j by

$$\exists\, C_1, C_2 > 0 : \quad j^*(y) \geq C_1|x| - C_2 \quad \text{if } y = \ell(x) \tag{1.31}$$

(see subsequent (2.H2) and Remark 2.2).

The main difficulties attached to the analysis of the PDE system (1.22)–(1.30) are related to the singular character of the entropy equations (1.22) and (1.24), to the nonlinear coupling between the latter equations, as well as between (1.28) and (1.29), and, last but not least, to the presence of *general* multivalued operators in all of the latter equations. In particular, it seems to us that dealing with a general maximal monotone operator in (1.22) and (1.24) brings about some technical difficulties, particularly in connection with the third type boundary condition (1.23) for ϑ on Γ_c.

All of these peculiarities will be carefully handled in the proof of our main result, Theorem 1 (see Sec. 2.3), stating the existence of *global in time* solutions to the Cauchy problem for (the generalized version of) system (1.22)–(1.30). We sketch below the main steps of our procedure, based on a suitable approximation of Problem (**P**), and on the derivation of suitable a priori estimates, which enable us to pass to the limit in the approximation. Such estimates are intrinsically related to the dissipative character of the system, highlighted in Remark 1.1. In fact, in the paper [6] we take advantage of the dissipative character of the system to perform its long-time analysis, showing that in the limit we reach a stationary equilibrium in which dissipation vanishes.

Note, however, that uniqueness is still an open issue, at least in the functional framework of our existence theorem. Without going into details, we may point out that the major obstacle is due to the singular character of equations (1.22) and (1.24). In particular, the boundary condition (1.23) makes it harder to prove contraction estimates leading to uniqueness. We refer to Remark 3.2 for additional observations on this point. Actually, uniqueness holds in the (more regular) framework of the approximate problem, see Section 3.5.

Plan of the paper. In Section 2, we enlist our assumptions on the problem data, present the variational formulation of the Cauchy problem for (a generalized version of) system (1.22)–(1.30), and state of our main result. In Section 3, we set up the approximation of Problem (**P**), suitably regularizing the maximal monotone operators in equations (1.22) and (1.24) and therein inserting (vanishing) viscosity terms. Hence, we prove a well-posedness result for the approximate problem. We combine a Schauder fixed point technique for local existence with a prolongation argument, based on global in time *a priori* estimates, while uniqueness follows from contraction estimates. Next, in Section 4 we pass to the limit in the approximate problem by compactness and monotonicity tools, and show that the approximate solutions converge to a solution of Problem (**P**). Finally, in the Appendix we prove some auxiliary technical results.

2 Main result

2.1 Setup and preliminary results

Notation 2.1. Throughout the paper, given a Banach space X, we denote by ${}_{X'}\langle\cdot,\cdot\rangle_X$ the duality pairing between X' and X itself, and by $\|\cdot\|_X$ both the norm in X and in any power of X; $C^0_{\mathrm{w}}([0,T];X)$ is the space of weakly continuous X-valued functions on $[0,T]$. Whenever $X = Y_1\times\ldots\times Y_N$, we denote by π_i, $i=1,\ldots,N$ the projection on the i-th component.

Young inequalities. We recall the Young inequality for convolutions, namely

$$\begin{aligned}&\forall\, p,\, q,\, r\in[1,\infty] \text{ s.t } \frac{1}{r}=\frac{1}{p}+\frac{1}{q}-1 \quad \forall\, a\in L^p(0,T)\ b\in L^q(0,T;X) \quad \text{we have}\\ &\qquad a*b\in L^r(0,T;X) \quad \text{and} \quad \|a*b\|_{L^r(0,T;X)}\le\|a\|_{L^p(0,T)}\|b\|_{L^q(0,T;X)}\,,\end{aligned} \tag{2.1}$$

and the Young inequality

$$\forall\,\delta>0\ \exists\, C_\delta>0: \quad \forall\, p,\, q\in(1,\infty) \text{ with } \frac{1}{p}+\frac{1}{q}=1 \ \ ab\le\delta a^p+C_\delta b^q \qquad \text{for all } a,\, b\in\mathbb{R}\,. \tag{2.2}$$

Functional setup. Henceforth, we shall suppose that Ω is a bounded smooth set of $\mathbb{R}^3$, such that Γ_c is a smooth bounded domain of $\mathbb{R}^2$, and use the notation

$$\begin{aligned}&H:=L^2(\Omega),\quad V:=H^1(\Omega),\quad \text{and}\\ &\mathbf{W}:=\left\{\mathbf{v}\in V^3\,:\,\mathbf{v}=\mathbf{0} \text{ a.e. on } \Gamma_1\right\}\,,\end{aligned}$$

the latter space endowed with the norm induced by V. We shall work with the standard Riesz operator

$$\mathcal{R}:V\to V' \ \text{ given by } \ {}_{V'}\langle\mathcal{R}(u),v\rangle_V:=\int_\Omega uv+\int_\Omega\nabla u\nabla v \quad \text{for all } u,\, v\in V, \tag{2.3}$$

and denote by $\mathcal{R}_{\Gamma_c}$ the analogously defined Riesz operator mapping $H^1(\Gamma_c)$ into $(H^1(\Gamma_c))'$. Further, we shall extensively use that

$$V\subset L^p(\Gamma_c) \ \text{ with a continuous (compact) embedding for } \ 1\le p\le4 \ \ (1\le p<4, \text{ resp.}), \tag{2.4}$$

$$H^1(\Gamma_c)\subset L^p(\Gamma_c) \ \text{ with a compact embedding for } \ 1\le p<\infty. \tag{2.5}$$

For simplicity, we denote by $\int_{\Gamma_c}uv$ ($\int_{\Gamma_2}uv$, resp.) the duality pairing ${}_{(H^{-1/2}(\Gamma_c))^3}\langle u,v\rangle_{(H^{1/2}(\Gamma_c))^3}$ between $(H^{-1/2}(\Gamma_c))^3$ and $(H^{1/2}(\Gamma_c))^3$ (between $(H^{-1/2}(\Gamma_2))^3$ and $(H^{1/2}(\Gamma_2))^3$, resp.). Finally, given a subset $\mathcal{O}\subset\mathbb{R}^N$, $N=1,2,3$, we shall denote by $|\mathcal{O}|$ its Lebesgue measure.

Preliminaries of viscoelasticity theory. We now introduce the standard bilinear forms of linear viscoelasticity which allow us to give a variational formulation of equation (1.26). Dealing with an anisotropic and inhomogeneous material, we assume that the fourth-order tensors $K=(a_{ijkh})$ and $K_v=(b_{ijkh})$, denoting the elasticity and the viscosity tensor, respectively, satisfy the classical symmetry and ellipticity conditions

$$\begin{aligned}&a_{ijkh}=a_{jikh}=a_{khij}\,,\qquad b_{ijkh}=b_{jikh}=b_{khij}\,,\quad i,j,k,h=1,2,3\\ &\exists\,\alpha_0>0: \qquad a_{ijkh}\xi_{ij}\xi_{kh}\ge\alpha_0\xi_{ij}\xi_{ij} \quad \forall\,\xi_{ij}:\xi_{ij}=\xi_{ji}\,,\quad i,j=1,2,3\,,\\ &\exists\,\beta_0>0: \qquad b_{ijkh}\xi_{ij}\xi_{kh}\ge\beta_0\xi_{ij}\xi_{ij} \quad \forall\,\xi_{ij}:\xi_{ij}=\xi_{ji}\,,\quad i,j=1,2,3\,,\end{aligned}$$

where the usual summation convention is used. Moreover, we require

$$a_{ijkh}, b_{ijkh} \in L^\infty(\Omega)\,, \quad i,j,k,h = 1,2,3.$$

By the previous assumptions on the elasticity and viscosity coefficients, the following bilinear forms $a, b : \mathbf{W} \times \mathbf{W} \to \mathbb{R}$, defined by

$$a(\mathbf{u},\mathbf{v}) := \int_\Omega a_{ijkh}\varepsilon_{kh}(\mathbf{u})\varepsilon_{ij}(\mathbf{v}) \quad \forall \mathbf{u},\mathbf{v} \in \mathbf{W},$$
$$b(\mathbf{u},\mathbf{v}) := \int_\Omega b_{ijkh}\varepsilon_{kh}(\mathbf{u})\varepsilon_{ij}(\mathbf{v}) \quad \forall \mathbf{u},\mathbf{v} \in \mathbf{W}$$

turn out to be continuous and symmetric. In particular, we have

$$\exists\, M > 0 : \ |a(\mathbf{u},\mathbf{v})| + |b(\mathbf{u},\mathbf{v})| \leq M \|\mathbf{u}\|_{\mathbf{W}} \|\mathbf{v}\|_{\mathbf{W}} \quad \forall \mathbf{u},\mathbf{v} \in \mathbf{W}. \tag{2.6}$$

Moreover, since Γ_1 has positive measure, by Korn's inequality we deduce that $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are $\mathbf{W}$-elliptic, i.e., there exist $C_a, C_b > 0$ such that

$$a(\mathbf{u},\mathbf{u}) \geq C_a \|\mathbf{u}\|^2_{\mathbf{W}} \qquad \forall\, \mathbf{u} \in \mathbf{W}, \tag{2.7}$$
$$b(\mathbf{u},\mathbf{u}) \geq C_b \|\mathbf{u}\|^2_{\mathbf{W}} \qquad \forall\, \mathbf{u} \in \mathbf{W}. \tag{2.8}$$

Relying on Green's formula (see, e.g., [18]), the variational formulation of (1.26) (cf. (2.36) below) can be derived by a standard argument.

2.2 Statement of the assumptions

As we mentioned in the introduction, we shall address the analysis of a *generalized version* of system (1.22)–(1.30), in which the operators occurring in (1.27) and (1.29) are replaced by general maximal monotone operators. We now enlist our assumptions on the involved nonlinearities and on the problem data.

We consider

$$\text{a proper, convex, and l.s.c. function } j : \mathbb{R} \to (-\infty, +\infty]\,, \tag{2.H1}$$

and

$$\text{its subdifferential in the sense of convex analysis } \ell = \partial j : \mathbb{R} \to 2^{\mathbb{R}}\,. \tag{2.9}$$

The crucial assumption on j is that the following *coercivity* condition holds:

$$\exists\, C_1,\ C_2 > 0 \quad \forall\, x \in \mathrm{D}(\ell),\ y \in \ell(x)\ : \quad yx - j(x) \geq C_1|x| - C_2\,. \tag{2.H2}$$

Remark 2.2. Note that (2.H2) can be rephrased as

$$\exists\, C_1,\ C_2 > 0 \quad \forall\, x \in \mathrm{D}(\ell),\ y \in \ell(x)\ : \quad j^*(y) \geq C_1|x| - C_2\,. \tag{2.10}$$

In particular, (2.10) is fulfilled when $j(x) = x(\log(x) - 1)$ for all $x \in (0, +\infty)$ and hence ℓ is the logarithmic nonlinearity, i.e. $\ell(x) = \log(x)$ for all $x \in (0, +\infty)$. In this case, simple computations show that $j^*(y) = e^y = \ell^{-1}(y)$ for all $y \in \mathbb{R}$, whence (2.10).

Henceforth, we shall denote by

$$\gamma \text{ the inverse of the operator } \ell \text{ (recall that } \gamma = \partial j^*)$$

and, with a slight abuse of notation, we shall call ℓ as well the realization of (2.9) as a maximal monotone operator $\ell : L^2(0,T;H) \to 2^{L^2(0,T;H)}$ (as a maximal monotone operator $\ell : L^2(0,T;L^2(\Gamma_c)) \to 2^{L^2(0,T;L^2(\Gamma_c))}$, respectively).

Assumptions on the other problem nonlinearities. Further, we let

$$\begin{gathered}\widehat{\alpha} : (H^{1/2}(\Gamma_c))^3 \to [0,+\infty] \text{ be a proper, convex and l.s.c. functional,}\\ \text{with } \widehat{\alpha}(\mathbf{0}) = 0 = \min \widehat{\alpha}, \text{ and}\\ \text{we set } \alpha := \partial\widehat{\alpha} : \ (H^{1/2}(\Gamma_c))^3 \to 2^{(H^{-1/2}(\Gamma_c))^3}.\end{gathered} \tag{2.H3}$$

Indeed, α shall generalize the subdifferential operator appearing in (1.28).

Remark 2.3. In fact, condition (1.28) may be rendered rigorously (see [4, Sec. II] for details) by introducing the set

$$\mathcal{X}_- := \{\mathbf{v} \in (H^{1/2}(\Gamma_c))^3 : \mathbf{v}\cdot\mathbf{n} \leq 0 \text{ a.e. in } \Gamma_c\},$$

with indicator function $I_{\mathcal{X}_-}$. Then we consider the maximal monotone operator $\partial I_{\mathcal{X}_-} : (H^{1/2}(\Gamma_c))^3 \to 2^{(H^{-1/2}(\Gamma_c))^3}$, given by

$$\begin{aligned}&\boldsymbol{\eta} \in (H^{-1/2}(\Gamma_c))^3 \text{ belongs to } \partial I_{\mathcal{X}_-}(\mathbf{y}) \text{ if and only if}\\ &\mathbf{y} \in \mathcal{X}_-, \quad \int_{\Gamma_c} \boldsymbol{\eta}\cdot(\mathbf{v}-\mathbf{y}) \leq 0 \quad \forall \mathbf{v} \in \mathcal{X}_-.\end{aligned}$$

In the same way, in equation (1.29) we shall consider

$$\text{a maximal monotone operator } \ \beta : \mathbb{R} \to 2^{\mathbb{R}}, \ \text{ with domain } \mathrm{D}(\beta) \subseteq [0,+\infty). \tag{2.H4}$$

It is well known that there exists a proper, l.s.c. and convex function $\widehat{\beta} : \overline{\mathrm{D}(\beta)} \to (-\infty,+\infty]$ such that $\beta = \partial\widehat{\beta}$. Concerning the functions λ, k, and σ', we assume that

$$\sigma' \ : \mathbb{R} \to \mathbb{R} \ \text{ is Lipschitz continuous, with Lipschitz constant } L_\sigma, \tag{2.H5}$$

$$k \ : \mathbb{R} \to [0,+\infty) \ \text{ is Lipschitz continuous , with Lipschitz constant } L_k, \tag{2.H6}$$

$$\lambda \in \mathrm{C}^{1,1}(\mathbb{R}) , \tag{2.H7}$$

i.e., λ has a Lipschitz continuous derivative. As a consequence,

$$\exists\, C_3 > 0 \ \forall\, x,y \in \mathbb{R} \ : \ \ |\lambda(x) - \lambda(y)| \leq C_3(|x|+|y|+1)|x-y| , \tag{2.11}$$

$$\exists\, C_3' > 0 \ \forall\, x \in \mathbb{R} \ : \ \ |\lambda'(x)| \leq C_3'(|x|+1) . \tag{2.12}$$

Assumptions on the problem data. We assume that

$$h \in L^2(0,T;V') \cap L^1(0,T;H) , \tag{2.H8}$$

$$\mathbf{f} \in L^2(0,T;H^3) , \tag{2.H9}$$

$$\mathbf{g} \in L^2(0,T;(H^{-1/2}(\Gamma_2))^3) . \tag{2.H10}$$

Then, we remark that the function $\mathbf{F} : (0,T) \to \mathbf{W}'$ defined by

$$_{\mathbf{W}'}\langle \mathbf{F}(t),\mathbf{v}\rangle_{\mathbf{W}} : \int_\Omega \mathbf{f}(t)\cdot\mathbf{v} + \int_{\Gamma_2} \mathbf{g}(t)\cdot\mathbf{v} \quad \forall\, \mathbf{v} \in W \quad \text{for a.e. } t \in (0,T),$$

fulfils

$$\mathbf{F} \in L^2(0,T;\mathbf{W}'). \tag{2.13}$$

Finally, we require that the initial data fulfil

$$w_0 \in H \ \text{ with } \ j^*(w_0) \in L^1(\Omega) , \tag{2.14}$$

$$z_0 \in L^2(\Gamma_c) \ \text{ and } \ j^*(z_0) \in L^1(\Gamma_c) , \tag{2.15}$$

$$\mathbf{u}_0 \in \mathbf{W} \ \text{ and } \ \mathbf{u}_0 \in \mathrm{D}(\widehat{\alpha}) , \tag{2.16}$$

$$\chi_0 \in H^1(\Gamma_c) , \ \ \widehat{\beta}(\chi_0) \in L^1(\Gamma_c) . \tag{2.17}$$

2.3 Variational formulation and statement of the main result

We now state the variational formulation of the initial-boundary value problem for a generalized version of system (1.22)–(1.30), featuring the nonlinearities introduced above.

Problem (P). Given a quadruple of initial data $(w_0, z_0, \mathbf{u}_0, \chi_0)$ complying with (2.14)–(2.17), find $(\vartheta, w, \vartheta_s, z, \mathbf{u}, \chi, \boldsymbol{\eta}, \xi)$ such that

$$\vartheta \in L^2(0,T;V) \cap L^\infty(0,T;L^1(\Omega)), \tag{2.18}$$
$$w \in L^\infty(0,T;H) \cap H^1(0,T;V'), \tag{2.19}$$
$$j^*(w) \in L^\infty(0,T;L^1(\Omega)), \tag{2.20}$$
$$\vartheta_s \in L^2(0,T;H^1(\Gamma_c)) \cap L^\infty(0,T;L^1(\Gamma_c)), \tag{2.21}$$
$$z \in L^\infty(0,T;L^2(\Gamma_c)) \cap H^1(0,T;H^1(\Gamma_c)'), \tag{2.22}$$
$$j^*(z) \in L^\infty(0,T;L^1(\Gamma_c)), \tag{2.23}$$
$$\mathbf{u} \in H^1(0,T;\mathbf{W}), \tag{2.24}$$
$$\boldsymbol{\eta} \in L^2(0,T;(H^{-1/2}(\Gamma_c))^3), \tag{2.25}$$
$$\chi \in L^2(0,T;H^2(\Gamma_c)) \cap L^\infty(0,T;H^1(\Gamma_c)) \cap H^1(0,T;L^2(\Gamma_c)), \tag{2.26}$$
$$\xi \in L^2(0,T;L^2(\Gamma_c)), \tag{2.27}$$

fulfilling the initial conditions

$$w(0) = w_0 \quad \text{a.e. in } \Omega, \tag{2.28}$$
$$z(0) = z_0 \quad \text{a.e. in } \Gamma_c, \tag{2.29}$$
$$\chi(0) = \chi_0 \quad \text{a.e. in } \Gamma_c, \tag{2.30}$$
$$\mathbf{u}(0) = \mathbf{u}_0 \quad \text{a.e. in } \Omega, \tag{2.31}$$

and

$$\begin{aligned} {}_{V'}\langle w_t, v\rangle_V - \int_\Omega \operatorname{div}(\mathbf{u}_t)\, v + \int_\Omega \nabla\vartheta\, \nabla v + \int_{\Gamma_c} k(\chi)(\vartheta - \vartheta_s)v \\ = {}_{V'}\langle h, v\rangle_V \quad \forall\, v \in V \ \text{ a.e. in } (0,T), \end{aligned} \tag{2.32}$$

$$w(x,t) \in \ell(\vartheta(x,t)) \quad \text{for a.e. } (x,t) \in \Omega \times (0,T), \tag{2.33}$$

$$\begin{aligned} {}_{H^1(\Gamma_c)'}\langle z_t, v\rangle_{H^1(\Gamma_c)} - \int_{\Gamma_c} \partial_t\lambda(\chi)\, v + \int_{\Gamma_c} \nabla\vartheta_s\, \nabla v \\ = \int_{\Gamma_c} k(\chi)(\vartheta - \vartheta_s)v \quad \forall\, v \in H^1(\Gamma_c) \ \text{ a.e. in } (0,T), \end{aligned} \tag{2.34}$$

$$z(x,t) \in \ell(\vartheta_s(x,t)) \quad \text{for a.e. } (x,t) \in \Gamma_c \times (0,T), \tag{2.35}$$

$$\begin{aligned} b(\mathbf{u}_t, \mathbf{v}) + a(\mathbf{u}, \mathbf{v}) + \int_\Omega \vartheta \operatorname{div}(\mathbf{v}) + \int_{\Gamma_c} (\chi\mathbf{u} + \boldsymbol{\eta}) \cdot \mathbf{v} \\ = {}_{\mathbf{W}'}\langle \mathbf{F}, \mathbf{v}\rangle_{\mathbf{W}} \quad \forall \mathbf{v} \in \mathbf{W} \ \text{ a.e. in } (0,T), \end{aligned} \tag{2.36}$$

$$\boldsymbol{\eta} \in \alpha(\mathbf{u}) \ \text{ in } (H^{-1/2}(\Gamma_c))^3 \quad \text{a.e. in } (0,T), \tag{2.37}$$

$$\chi_t - \Delta\chi + \xi + \sigma'(\chi) = -\lambda'(\chi)\vartheta_s - \frac{1}{2}|\mathbf{u}|^2 \quad \text{a.e. in } \Gamma_c \times (0,T), \tag{2.38}$$

$$\xi \in \beta(\chi) \quad \text{a.e. in } \Gamma_c \times (0,T), \tag{2.39}$$

$$\partial_{\mathbf{n}_s}\chi = 0 \text{ a.e. in } \partial\Gamma_c \times (0,T). \tag{2.40}$$

Note that, to simplify notation, we have incorporated the contribution $-\lambda'(\chi)\vartheta_{\mathrm{eq}}$ occurring in (1.29) into the term $\sigma'(\chi)$ in (2.38).

Theorem 1 (Existence of a global solution)**.** *Assume* (2.H1)–(2.H10)*. Then, Problem* (**P**) *admits at least a global solution* $(\vartheta, w, \vartheta_s, z, \mathbf{u}, \chi, \boldsymbol{\eta}, \xi)$ *with the regularity* (2.18)–(2.27)*.*

Remark 2.4 (Positivity of the temperature)**.** Clearly, in the case

$$\mathrm{D}(j) \subseteq (0, +\infty),$$

(such as for (1.2)), from relations (2.33) and (2.35) we infer that both system temperatures ϑ and ϑ_s are strictly positive almost everywhere in $\Omega \times (0,T)$ and in $\Gamma_c \times (0,T)$, respectively.

Strategy of the proof of Theorem 1. We shall approximate Problem (**P**) by suitably regularizing equations (2.32) and (2.34). More precisely, we shall add some viscosity terms to both equations, and replace the operator ℓ therein by a Yosida-type regularization (see (3.1) below). For technical reasons (cf. with Remark 3.2), we shall keep the viscosity parameter distinct from the Yosida regularization parameter, denoting by $\varepsilon > 0$ the former and by $\mu > 0$ the latter. Hence, we shall call $(\mathbf{P}_\varepsilon^\mu)$ the initial and boundary value problem for the resulting approximate system and prove that it is well-posed following this outline: first in Sections 3.2–3.3 we are going to prove the existence of a *local solution* by a fixed point argument. Next, in Section 3.4 we are going to extend such a solution to the whole interval $(0,T)$, while in Section 3.5 we shall obtain contraction estimates leading to uniqueness for Problem $(\mathbf{P}_\varepsilon^\mu)$.

Finally, in order to prove Theorem 1 we shall pass to the limit in Problem $(\mathbf{P}_\varepsilon^\mu)$ in two steps. First, we shall keep $\mu > 0$ fixed and let $\varepsilon \searrow 0$: in Section 4.1 we are going to show that the approximate solutions converge, as $\varepsilon \searrow 0$, to a solution of the initial and boundary value problem obtained by setting $\varepsilon = 0$ in the approximate equations (3.13)–(3.14) below. Secondly, we shall also let $\mu \searrow 0$ and obtain in the limit a solution to Problem (**P**).

Notation 2.5. Henceforth, for the sake of notational simplicity, we shall use the same symbol $\langle \cdot, \cdot \rangle$ for the duality pairings ${}_{\mathbf{W}'}\langle \cdot, \cdot \rangle_{\mathbf{W}}$, ${}_{V'}\langle \cdot, \cdot \rangle_{V}$, and ${}_{H^1(\Gamma_c)'}\langle \cdot, \cdot \rangle_{H^1(\Gamma_c)}$, and, further, denote by

$$\text{the symbols } C,\ C' \text{ most of the (positive) constants} \tag{2.41}$$

occurring in calculations and estimates.

3 Approximation

3.1 The approximate problem

We approximate Problem (**P**) by modifying equations (2.32) and (2.34) in the following way:

- first, we shall add to (2.32) the regularizing viscosity term $\varepsilon\mathcal{R}(\vartheta_t)$ and to (2.34) the viscosity term $\varepsilon\mathcal{R}_{\Gamma_c}(\partial_t\vartheta_s)$, with $\varepsilon > 0$: this shall enable us to perform enhanced regularity estimates on the (approximate) equations for the temperatures ϑ and ϑ_s and ultimately to prove the global well-posedness of the approximate system, see also Remark 3.2;
- second, both in (2.32) and in (2.34) we shall replace the operator ℓ with its Yosida-type regularization
$$\mathscr{L}_\mu := (\mu \mathrm{Id} + \gamma_\mu)^{-1}, \tag{3.1}$$
where $\mu > 0$, $\mathrm{Id} : \mathbb{R} \to \mathbb{R}$ is the identity function, and $\gamma_\mu : \mathbb{R} \to \mathbb{R}$ is the (μ-)Yosida regularization of the inverse γ of ℓ, see (3.2) below. The choice of $\mathscr{L}_\mu$ is motivated by technical reasons, cf. with Remarks 3.2 and 4.2 later on.

For the purposes of the above approximation, we recall that the Yosida regularization $\gamma_\mu : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function, given by

$$\gamma_\mu(w) := \frac{1}{\mu}\left(w - \rho_\mu(w)\right) \qquad \text{for all } w \in \mathbb{R}, \tag{3.2}$$

where $\rho_\mu : \mathbb{R} \to \mathbb{R}$ is the (μ-)resolvent operator associated with γ, defined for every $w \in \mathbb{R}$ as the unique solution $\rho_\mu(w)$ of the inclusion

$$\rho_\mu(w) - w + \mu\gamma(\rho_\mu(w)) \ni 0. \tag{3.3}$$

We shall also deal with the Yosida approximation j^*_μ of j^*, defined for every $\mu > 0$ by

$$j^*_\mu(w) := \min_{y\in\mathbb{R}} \left\{ \frac{|y-w|^2}{2\mu} + j^*(w) \right\} \qquad \text{for all } w \in \mathbb{R}. \tag{3.4}$$

We recall that $j^*_\mu \in C^1(\mathbb{R})$, with derivative $j^{*\prime}_\mu = \gamma_\mu$, and that it fulfils

$$j^*_\mu(w) = \frac{\mu}{2}|\gamma_\mu(w)|^2 + j^*\left(\rho_\mu(w)\right) \qquad \forall\, w \in \mathbb{R}\,. \tag{3.5}$$

It was proved in [13, Sec. 3] that the operator $\mathscr{L}_\mu$ is well-defined on $\mathbb{R}$, monotone, and that

$$\mathscr{L}_\mu \colon \mathbb{R} \to \mathbb{R} \ \text{ is Lipschitz continuous with Lipschitz constant } 1/\mu. \tag{3.6}$$

Approximate initial data. In order to properly state our approximate problem, depending on the parameters $\varepsilon > 0$ and $\mu > 0$, we shall need to prescribe some initial conditions for ϑ and ϑ_s. To this aim, in Lemma 4.1 we shall construct sequences of initial data $\vartheta^0_{\varepsilon\mu}$ and $\vartheta^0_{s,\varepsilon\mu}$ satisfying

$$\vartheta^0_{\varepsilon\mu} \in V\,, \qquad j^*_\mu\left(\mathscr{L}_\mu(\vartheta^0_{\varepsilon\mu})\right) \in L^1(\Omega)\,, \tag{3.7}$$

$$\vartheta^0_{s,\varepsilon\mu} \in H^1(\Gamma_c)\,, \qquad j^*_\mu\left(\mathscr{L}_\mu(\vartheta^0_{s,\varepsilon\mu})\right) \in L^1(\Gamma_c)\,, \tag{3.8}$$

and such that there exists a constant $M_0 > 0$ *independent of* $\varepsilon > 0$ *and* $\mu > 0$ with

$$\left\|j^*_\mu\left(\mathscr{L}_\mu(\vartheta^0_{\varepsilon\mu})\right)\right\|_{L^1(\Omega)} + \left\|j^*_\mu\left(\mathscr{L}_\mu(\vartheta^0_{s,\varepsilon\mu})\right)\right\|_{L^1(\Gamma_c)} \le M_0 \quad \text{for all } \varepsilon,\ \mu > 0, \tag{3.9}$$

and a constant $M_0^\mu > 0$ *independent of* $\varepsilon > 0$ but possibly depending on $\mu > 0$ with

$$\varepsilon^{1/2}\|\vartheta^0_{\varepsilon\mu}\|_V + \varepsilon^{1/2}\|\vartheta^0_{s,\varepsilon\mu}\|_{H^1(\Gamma_c)} \le M_0^\mu \quad \text{for all } \varepsilon > 0. \tag{3.10}$$

For simplicity, throughout this section we shall omit the indexes ε and μ in the notation for the initial data. The variational formulation of the initial-boundary value problem approximating Problem (**P**) then reads:

Problem ($\mathbf{P}^\mu_\varepsilon$). Given a quadruple of initial data $(\vartheta^0, \vartheta^0_s, \mathbf{u}_0, \chi_0)$ fulfilling (3.7)–(3.10) and (2.16)–(2.17), find functions $(\vartheta, \vartheta_s, \mathbf{u}, \chi, \xi, \boldsymbol{\eta})$ satisfying (2.24)–(2.27), with ϑ and ϑ_s such that

$$\vartheta \in H^1(0,T;V)\,, \tag{3.11}$$

$$\vartheta_s \in H^1(0,T;H^1(\Gamma_c))\,, \tag{3.12}$$

fulfilling

$$\begin{aligned}\varepsilon \int_\Omega \vartheta_t\, v + \int_\Omega \partial_t \mathscr{L}_\mu(\vartheta) v - \int_\Omega \mathrm{div}(\mathbf{u}_t)\, v + \varepsilon \int_\Omega \nabla \vartheta_t \nabla v + \int_\Omega \nabla \vartheta \nabla v \\ + \int_{\Gamma_c} k(\chi)(\vartheta - \vartheta_s) v = \langle h, v\rangle \quad \forall\, v \in V \quad \text{a.e. in } (0,T)\,,\end{aligned} \tag{3.13}$$

$$\begin{aligned}\varepsilon \int_{\Gamma_c} \partial_t \vartheta_s\, v + \int_{\Gamma_c} \partial_t \mathscr{L}_\mu(\vartheta_s) v + \varepsilon \int_{\Gamma_c} \nabla \partial_t \vartheta_s \nabla v \\ - \int_{\Gamma_c} \partial_t \lambda(\chi)\, v + \int_{\Gamma_c} \nabla \vartheta_s \nabla v = \int_{\Gamma_c} k(\chi)(\vartheta - \vartheta_s) v \quad \forall\, v \in H^1(\Gamma_c) \quad \text{a.e. in } (0,T)\,,\end{aligned} \tag{3.14}$$

and equations (2.36)–(2.40), along with the initial conditions (2.31) and (2.30) for $\mathbf{u}$ and χ, respectively, and, for ϑ and ϑ_s,

$$\vartheta(0) = \vartheta^0 \qquad \text{in } V\,, \tag{3.15}$$

$$\vartheta_s(0) = \vartheta_s^0 \qquad \text{in } H^1(\Gamma_c)\,. \tag{3.16}$$

Remark 3.1. Combining (3.11)–(3.12) with the fact that $\mathscr{L}_\mu$ is Lipschitz continuous (cf. (3.6)), we conclude that for any solution $(\vartheta, \vartheta_s, \mathbf{u}, \chi, \xi, \boldsymbol{\eta})$ there holds

$$\mathscr{L}_\mu(\vartheta) \in L^\infty(0,T;V) \cap H^1(0,T;H), \qquad \mathscr{L}_\mu(\vartheta_s) \in L^\infty(0,T;H^1(\Gamma_c)) \cap H^1(0,T;L^2(\Gamma_c))\,. \tag{3.17}$$

Remark 3.2. Without going into details, a few comments on the approximate Problem $(\mathbf{P}_\varepsilon^\mu)$ are in order. Besides regularizing the maximal monotone operators in (2.32) and in (2.34), we have inserted in both equations a viscosity term in order to make each of (the Cauchy problems for) the approximate equations well-posed. To understand why, let us focus on equation (3.13) (analogous considerations apply to equation (3.14)).

Indeed, because of the nonlinear term $\partial_t(\mathscr{L}_\mu(\vartheta))$ in (3.13), in order to prove that the (Cauchy problem for the) latter equation has a unique solution, one has to integrate it in time. Hence, as it will be clear from the calculations in Lemma 3.5, the additional viscosity term $\varepsilon\mathcal{R}(\vartheta_t)$ enables us to deal, in the integrated version of (3.13), with the third type boundary condition on ϑ. On the other hand, the choice of the operator $\mathscr{L}_\mu$ in (3.13) (of $\mathscr{L}_\mu$ in (3.14)), in place of the usual Yosida regularization of ℓ, is due to technical reasons connected to the construction of sequences of approximate initial data fulfilling (3.7)–(3.10), see Lemma 4.1 and Remark 4.2 later on.

The ultimate reason why we keep the viscosity parameter ε distinct from the Yosida parameter μ in both approximate equations (2.32) and (2.34) is due to the fact that, in order to recover the $L^\infty(0,T;H)$-regularity (2.19) for the solution component w (the $L^\infty(0,T;L^2(\Gamma_c))$-regularity (2.22) for the solution component z, respectively), one has to test some approximation of (2.32) ((2.34), respectively) by a function approximating w (z, resp.), and obtain some bound obviously independent of the approximation parameter. In the present framework, performing such an estimate on equation (3.13) (on (3.14), resp.) with $\varepsilon > 0$ would not lead to estimates on $\mathscr{L}_\mu(\vartheta)$ independent of the parameters ε and μ, essentially because the term $\langle \varepsilon\mathcal{R}(\vartheta_t), \mathscr{L}_\mu(\vartheta)\rangle$ ($\langle \varepsilon\mathcal{R}_{\Gamma_c}(\partial_t\vartheta_s), \mathscr{L}_\mu(\vartheta_s)\rangle$, resp.) cannot be dealt with by monotonicity arguments. That is why, in Section 4.2 we shall perform the crucial estimate leading to regularity (2.19) and (2.22) only after taking the limit in Problem $(\mathbf{P}_\varepsilon^\mu)$ as $\varepsilon \searrow 0$ with $\mu > 0$ fixed.

The following theorem holds.

Theorem 3.1 (Global well-posedness for Problem $(\mathbf{P}_\varepsilon^\mu)$)**.** *Under assumptions* (2.H1)–(2.H10), *for any set of initial data* $(\vartheta^0, \vartheta_s^0, \mathbf{u}_0, \chi_0)$ *complying with conditions* (3.7)–(3.10) *and* (2.16)–(2.17) *and for all* $\varepsilon,\ \mu > 0$ *there exists a unique global solution* $(\vartheta, \vartheta_s, \mathbf{u}, \chi, \xi, \boldsymbol{\eta})$ *to Problem* $(\mathbf{P}_\varepsilon^\mu)$.

As already mentioned, we shall first of all prove the existence of a local solution to Problem ($\mathbf{P}_\varepsilon^\mu$) by means of a Schauder fixed point argument, which relies on auxiliary intermediate results on the existence and uniqueness of solutions for the single equations of ($\mathbf{P}_\varepsilon^\mu$). We shall prove such results in the following Section 3.2, and conclude the proof of local existence in Section 3.3. Finally, in Section 3.4 we shall show that the local solution extends to a (unique, by Section 3.5) global one.

3.2 Fixed point setup

For a fixed $\mathsf{t} \in (0, T]$ (which shall be specified later on) and a fixed constant $R > 0$, we consider the set

$$\mathcal{S}_\mathsf{t} := \Big\{ (\vartheta, \vartheta_s, \chi) \in L^2(0,\mathsf{t}; H) \times L^{5/2}(0,\mathsf{t}; L^3(\Gamma_c)) \times L^{10}(0,\mathsf{t}; L^6(\Gamma_c)) \ : \\ \|(\vartheta, \vartheta_s, \chi)\|_{L^2(0,\mathsf{t};H)\times L^{5/2}(0,\mathsf{t};L^3(\Gamma_c))\times L^{10}(0,\mathsf{t};L^6(\Gamma_c))} \leq R \Big\}. \tag{3.18}$$

We shall construct an operator $\mathcal{T}$ (its definition is split in several steps), mapping $\mathcal{S}_{\widehat{T}}$ into itself for a suitable time $0 < \widehat{T} \leq T$, in such a way that any fixed point of $\mathcal{T}$ yields a solution to Problem ($\mathbf{P}_\varepsilon^\mu$) on the interval $(0, \widehat{T})$. Then, in Proposition 3.7 we shall prove that $\mathcal{T} : \mathcal{S}_{\widehat{T}} \to \mathcal{S}_{\widehat{T}}$ admits a fixed point by the Schauder theorem.

Notation 3.3. We shall denote by

$$\begin{gathered} M_i, \ i = 1,2,3 \ \text{ a positive constant depending on } R, \text{ on the problem data,} \\ \text{on } M_0 \text{ (cf. (3.9)) and } M_0^\mu \text{ (cf. (3.10)),} \\ \text{but } \textit{independent} \text{ of } \varepsilon > 0, \text{ and of the fixed } \ \mathsf{t} \in (0,T] \\ M_i^\varepsilon, \ i = 1,2,3 \ \text{ a positive constant depending on } R, \text{ on the problem data,} \\ \text{on } M_0 \text{ (cf. (3.9)) and } M_0^\mu \text{ (cf. (3.10)), and possibly on } \varepsilon > 0, \\ \text{but in any case } \textit{independent} \text{ of the fixed } \ \mathsf{t} \in (0,T]\,. \end{gathered}$$

Further, we shall keep to the notation (2.41) for all the constants which do not depend on the approximating parameter ε.

Step 1. We take $(\widehat{\vartheta}, \widehat{\vartheta}_s, \widehat{\chi}) \in \mathcal{S}_\mathsf{t}$ and consider (the Cauchy problem) for system (2.36)–(2.40), in which ϑ in (2.36) is replaced by $\widehat{\vartheta}$ and χ and ϑ_s on the right-hand side of (2.38) are replaced by $\widehat{\chi}$ and $\widehat{\vartheta}_s$, respectively.

Lemma 3.4. *Assume* (2.H3), (2.H4), (2.H5), (2.H7), (2.H9), (2.H10), (2.16), *and* (2.17).
Then, there exists a constant $M_1 > 0$ *such that for all* $(\widehat{\vartheta}, \widehat{\vartheta}_s, \widehat{\chi}) \in \mathcal{S}_\mathsf{t}$ *there exists a unique quadruple* $(\mathbf{u}, \boldsymbol{\eta}, \chi, \xi)$, *with the regularity*

$$\|\mathbf{u}\|_{H^1(0,\mathsf{t};\mathbf{W})} + \|\boldsymbol{\eta}\|_{L^2(0,\mathsf{t};H^{-1/2}(\Gamma_c))} + \|\xi\|_{L^2(0,\mathsf{t};L^2(\Gamma_c))} \\ + \|\chi\|_{L^2(0,\mathsf{t};H^2(\Gamma_c))\cap L^\infty(0,\mathsf{t};H^1(\Gamma_c))\cap H^1(0,\mathsf{t};L^2(\Gamma_c))} \leq M_1\,, \tag{3.19}$$

complying with the initial conditions (2.30)–(2.31), *solving the PDE system*

$$\begin{aligned} & b(\mathbf{u}_t, \mathbf{v}) + a(\mathbf{u}, \mathbf{v}) + \int_{\Gamma_c} (\chi \mathbf{u} + \boldsymbol{\eta}) \cdot \mathbf{v} \\ & = \langle \mathbf{F}, \mathbf{v} \rangle - \int_\Omega \widehat{\vartheta} \, \mathrm{div}(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{W} \quad \textit{a.e. in } (0,\mathsf{t})\,, \\ & \chi_t - \Delta\chi + \xi + \sigma'(\chi) = -\lambda'(\widehat{\chi})\widehat{\vartheta}_s - \frac{1}{2}|\mathbf{u}|^2 \quad \textit{a.e. in } \Gamma_c \times (0,\mathsf{t})\,, \end{aligned} \tag{3.20}$$

and such that $\mathbf{u}$ *and* $\boldsymbol{\eta}$ *fulfil* (2.37)*,* χ *and* ξ *comply with* (2.39)*–*(2.40) *on* $(0,\mathsf{t})$*.*

Let us point out that, in view of (2.H7), (3.18), and the Hölder inequality, there holds

$$\|\lambda'(\widehat{\chi})\,\widehat{\vartheta}_s\|_{L^2(0,\mathsf{t};L^2(\Gamma_c))} \leq C\|\widehat{\chi}\|_{L^{10}(0,\mathsf{t};L^6(\Gamma_c))}\,\|\widehat{\vartheta}_s\|_{L^{5/2}(0,\mathsf{t};L^3(\Gamma_c))} \leq CR^2\,. \tag{3.21}$$

Then, also taking into account the fact that $\widehat{\vartheta} \in L^2(0,\mathsf{t};L^2(\Gamma_c))$, Lemma 3.4 follows from [5, Thm. 1], to which we refer the reader.

Now, we let

$$\begin{aligned}\mathcal{V}_{\mathsf{t}} := \Big\{&(\mathbf{u},\chi) \in H^1(0,\mathsf{t};\mathbf{W}) \times \big(L^2(0,\mathsf{t};H^2(\Gamma_c)) \cap L^\infty(0,\mathsf{t};H^1(\Gamma_c)) \cap H^1(0,\mathsf{t};L^2(\Gamma_c))\big) \ : \\ &\|\mathbf{u}\|_{H^1(0,\mathsf{t};\mathbf{W})} + \|\chi\|_{L^2(0,\mathsf{t};H^2(\Gamma_c))\cap L^\infty(0,\mathsf{t};H^1(\Gamma_c))\cap H^1(0,\mathsf{t};L^2(\Gamma_c))} \leq M_1\Big\}\,.\end{aligned} \tag{3.22}$$

Thanks to Lemma 3.4, we may define an operator

$$\mathcal{T}_1 : \mathcal{S}_{\mathsf{t}} \to \mathcal{V}_{\mathsf{t}}$$

mapping every triple $(\widehat{\vartheta},\widehat{\vartheta}_s,\widehat{\chi}) \in \mathcal{S}_{\mathsf{t}}$ into the pair $(\mathbf{u},\chi)$ solving the Cauchy problem for system (3.20) (we imply that with $(\mathbf{u},\chi)$ the solution components $\boldsymbol{\eta}$ and ξ satisfying (2.37) and (2.39) are uniquely associated).

Step 2. We fix $(\widehat{\vartheta}_s,\overline{\mathbf{u}},\overline{\chi}) \in \pi_2(\mathcal{S}_{\mathsf{t}}) \times \mathcal{V}_{\mathsf{t}}$ and consider (the Cauchy problem) for (3.13) with data $(\widehat{\vartheta}_s,\overline{\mathbf{u}},\overline{\chi})$.

Lemma 3.5. *Assume* (2.H1)*,* (2.H2)*,* (2.H6)*,* (2.H8)*, and* (3.7)*–*(3.10)*.*

Then, there exist $M_2, M_2^\varepsilon > 0$ *such that for all* $(\widehat{\vartheta}_s,\overline{\mathbf{u}},\overline{\chi}) \in \pi_2(\mathcal{S}_{\mathsf{t}}) \times \mathcal{V}_{\mathsf{t}}$ *there exists a unique* ϑ *with*

$$\|\vartheta\|_{L^2(0,\mathsf{t};V)\cap L^\infty(0,\mathsf{t};L^1(\Omega))} \leq M_2, \quad \|\vartheta_t\|_{L^2(0,\mathsf{t};V)} \leq M_2^\varepsilon\,, \tag{3.23}$$

complying with the initial condition (3.15) *and with*

$$\begin{aligned}\varepsilon\int_\Omega \vartheta_t\, v + \int_\Omega \partial_t \mathscr{L}_\mu(\vartheta) v - \int_\Omega \mathrm{div}(\overline{\mathbf{u}}_t)\, v + \varepsilon\int_\Omega \nabla\vartheta_t\,\nabla v + \int_\Omega \nabla\vartheta\,\nabla v \\ + \int_{\Gamma_c} k(\overline{\chi})(\vartheta - \widehat{\vartheta}_s) v = \langle h, v\rangle \quad \forall\, v \in V \quad \textit{a.e. in } (0,\mathsf{t})\,.\end{aligned} \tag{3.24}$$

Proof. For simplicity, throughout the proof we shall use the notation

$$\mathcal{K}(x,t) := k(\overline{\chi}(x,t)) \qquad \text{for a.e. } (x,t) \in \Gamma_c \times (0,\mathsf{t})\,. \tag{3.25}$$

It follows from (2.H6), the regularity of $\overline{\chi}$ (cf. (3.22)), and (2.5) that

$$\mathcal{K} \in L^\infty(0,\mathsf{t};H^1(\Gamma_c))\,, \quad \text{hence} \quad \mathcal{K} \in L^\infty(0,\mathsf{t};L^p(\Gamma_c)) \quad \forall\, 1 \leq p < \infty\,. \tag{3.26}$$

In view of [17, Thm. 1], the Cauchy problem for (3.24) has at least a solution $\vartheta \in H^1(0,\mathsf{t};V)$. In order to prove uniqueness, we let ϑ_1, $\vartheta_2 \in H^1(0,\mathsf{t};V)$ be two solutions of the Cauchy problem (3.15, 3.24), and set $\widetilde{\vartheta} : \vartheta_1 - \vartheta_2$. We subtract the equation for ϑ_2 from the equation for ϑ_1 and integrate on $(0,t)$, with $0 \leq t \leq \mathsf{t}$. Thus, we get

$$\begin{aligned}\varepsilon\int_\Omega \widetilde{\vartheta}(t)\, v + \int_\Omega (\mathscr{L}_\mu(\vartheta_1(t)) - \mathscr{L}_\mu(\vartheta_2(t)))\, v + \varepsilon\int_\Omega \nabla\widetilde{\vartheta}(t)\,\nabla v \\ + \int_\Omega (1 * \nabla\widetilde{\vartheta})(t)\,\nabla v + \int_{\Gamma_c} (1 * \mathcal{K}\widetilde{\vartheta})(t)\, v = 0\end{aligned}$$

for all $v \in V$. Hence, we take $v = \widetilde{\vartheta}$ and integrate in time: also using that the operator $\mathscr{L}_\mu$ is monotone, with straightforward computations we obtain

$$\varepsilon \int_0^t \|\widetilde{\vartheta}\|_H^2 + \varepsilon \int_0^t \|\nabla\widetilde{\vartheta}\|_H^2 + \frac{1}{2}\|(1 * \nabla\widetilde{\vartheta})(t)\|_H^2 \leq \int_0^t \|\widetilde{\vartheta}\|_{L^2(\Gamma_c)} \|(1 * \mathcal{K}\widetilde{\vartheta})\|_{L^2(\Gamma_c)}. \tag{3.27}$$

In view of the Young inequality for convolutions (2.1), we have

$$\|(1 * \mathcal{K}\widetilde{\vartheta})(s)\|_{L^2(\Gamma_c)} \leq s^{1/2}\|\mathcal{K}\widetilde{\vartheta}\|_{L^2(0,s;L^2(\Gamma_c))} \leq s^{1/2}\|\mathcal{K}\|_{L^\infty(0,s;L^4(\Gamma_c))}\|\widetilde{\vartheta}\|_{L^2(0,s;L^4(\Gamma_c))} \quad \forall\, s \in [0,t]\,.$$

Hence, (3.27) yields

$$\varepsilon \int_0^t \|\widetilde{\vartheta}\|_V^2 \leq \frac{\varepsilon}{2} \int_0^t \|\widetilde{\vartheta}\|_V^2 + C_\varepsilon t \|\mathcal{K}\|^2_{L^\infty(0,\mathsf{t};H^1(\Gamma_c))} \int_0^t \|\widetilde{\vartheta}\|^2_{L^2(0,s;V)} \,\mathrm{d}s\,, \tag{3.28}$$

where C_ε depends on ε and also on the embedding constants in (2.4)–(2.5). By applying the Gronwall Lemma (see, e.g., [14, Lemma A.3]), we end up with

$$\widetilde{\vartheta}(t) = 0 \qquad \text{for a.e.}\, t \in (0,\mathsf{t})\,,$$

whence the desired uniqueness.

We prove estimate (3.23) by testing (3.24) by ϑ. Using the definition (3.1) of $\mathscr{L}_\mu$, we find

$$\begin{aligned} \int_\Omega \partial_t \mathscr{L}_\mu(\vartheta)\vartheta &= \mu \int_\Omega \partial_t \mathscr{L}_\mu(\vartheta)\mathscr{L}_\mu(\vartheta) + \int_\Omega \partial_t \mathscr{L}_\mu(\vartheta)\gamma_\mu(\mathscr{L}_\mu(\vartheta)) \\ &= \frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathscr{L}_\mu(\vartheta)\|_H^2 + \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega j_\mu^*(\mathscr{L}_\mu(\vartheta))\,, \end{aligned} \tag{3.29}$$

the latter inequality ensuing from the chain rule for the convex functional j_μ^*. Hence, upon integrating in time we get

$$\begin{aligned} &\frac{\varepsilon}{2}\|\vartheta(t)\|_V^2 + \frac{\mu}{2}\|\mathscr{L}_\mu(\vartheta(t))\|_H^2 + \int_\Omega j_\mu^*(\mathscr{L}_\mu(\vartheta(t))) + \int_0^t \|\nabla\vartheta\|_H^2 + \int_0^t \int_{\Gamma_c} \mathcal{K}\vartheta^2 \\ &\quad = \frac{\varepsilon}{2}\|\vartheta^0\|_V^2 + \frac{\mu}{2}\left\|\mathscr{L}_\mu(\vartheta^0)\right\|_H^2 + \int_\Omega j_\mu^*\left(\mathscr{L}_\mu(\vartheta^0)\right) + I_1 + I_2 + I_3\,, \end{aligned} \tag{3.30}$$

where the integral terms I_i, $i = 1,2,3$ are specified by (3.32)–(3.34) below. Now, in view of (2.H2) (see Lemma A.1 later on), there exist C_1, $\overline{C}_2 > 0$ not depending on $\varepsilon \in (0,1)$ such that

$$\mu\,\|\mathscr{L}_\mu(\vartheta(t))\|_H^2 + \int_\Omega j_\mu^*(\mathscr{L}_\mu(\vartheta(t))) \geq C_1\|\vartheta(t)\|_{L^1(\Omega)} - \overline{C}_2\,, \tag{3.31}$$

whereas by (3.7) and (3.9)–(3.10) we estimate the first three summands on the right-hand side of (3.30). Further, we estimate

$$\begin{aligned} I_1 &= \int_0^t \int_{\Gamma_c} \mathcal{K}\,\widehat{\vartheta}_s\,\vartheta \leq \int_0^t \|\mathcal{K}\|_{L^6(\Gamma_c)}\,\|\widehat{\vartheta}_s\|_{L^3(\Gamma_c)}\,\|\vartheta\|_{L^2(\Gamma_c)} \\ &\leq C\int_0^t \|\mathcal{K}\|_{H^1(\Gamma_c)}\,\|\widehat{\vartheta}_s\|_{L^3(\Gamma_c)}\,(\|\vartheta - m(\vartheta)\|_V + \|m(\vartheta)\|_V) \\ &\leq C\int_0^t \|\mathcal{K}\|_{H^1(\Gamma_c)}\,\|\widehat{\vartheta}_s\|_{L^3(\Gamma_c)}\,\|\nabla\vartheta\|_{L^2(\Omega)} + C\int_0^t \|\mathcal{K}\|_{H^1(\Gamma_c)}\,\|\widehat{\vartheta}_s\|_{L^3(\Gamma_c)}\,\|\vartheta\|_{L^1(\Omega)} \\ &\leq \rho\int_0^t \|\nabla\vartheta\|^2_{L^2(\Omega)} + C_\rho\|\mathcal{K}\|^2_{L^{10}(0,\mathsf{t};H^1(\Gamma_c))}\,\|\widehat{\vartheta}_s\|^2_{L^{5/2}(0,\mathsf{t};L^3(\Gamma_c))} \\ &\qquad + C\int_0^t \|\mathcal{K}\|_{H^1(\Gamma_c)}\,\|\widehat{\vartheta}_s\|_{L^3(\Gamma_c)}\,\|\vartheta\|_{L^1(\Omega)}\,, \end{aligned} \tag{3.32}$$

where the second inequality follows from (2.4)–(2.5), the third one from the Poincaré inequality, and, thanks to Young's inequality (2.2), the last one holds for a suitable $\rho > 0$ to be chosen later. In the same way, we estimate

$$
\begin{aligned}
I_2 &= \int_0^t \int_\Omega \mathrm{div}(\overline{\mathbf{u}}_t)\,\vartheta \le \int_0^t \|\,\mathrm{div}(\overline{\mathbf{u}}_t)\|_H \,\|\vartheta\|_H \\
&\le C \int_0^t \|\,\mathrm{div}(\overline{\mathbf{u}}_t)\|_H \,\left(\|\vartheta - m(\vartheta)\|_V + \|m(\vartheta)\|_V\right) \\
&\le \rho \int_0^t \|\nabla\vartheta\|^2_{L^2(\Omega)} + C_\rho{}' \int_0^t \|\overline{\mathbf{u}}_t\|^2_{\mathbf{W}} + C\int_0^t \|\overline{\mathbf{u}}_t\|_{\mathbf{W}}\|\vartheta\|_{L^1(\Omega)},
\end{aligned}
\tag{3.33}
$$

$$
\begin{aligned}
I_3 = \int_0^t \langle h, \vartheta\rangle &\le \int_0^t \|h\|_{V'}\,\|\vartheta\|_V \\
&\le \rho \int_0^t \|\nabla\vartheta\|^2_{L^2(\Omega)} + C_\rho{}'' \int_0^t \|h\|^2_{V'} + C\int_0^t \|h\|_{V'}\|\vartheta\|_{L^1(\Omega)}\,.
\end{aligned}
\tag{3.34}
$$

Now, collecting (3.30)–(3.31) and (3.32)–(3.34) (in which we choose $\rho \le 1/6$), taking into account (3.26), estimate (3.19) for $\overline{\mathbf{u}}$ and (3.18) for $\widehat{\vartheta}_s$, using that fourth term on the left-hand side of (3.30) is nonnegative thanks to (2.H6), and applying the Gronwall Lemma, we infer that there exists $M_2 > 0$, independent of ε and μ, such that

$$
\varepsilon^{1/2}\|\vartheta(t)\|_V + \|\vartheta\|_{L^2(0,t;V)} + \|\vartheta(t)\|_{L^1(\Omega)} \le M_2 \qquad \forall\, t \in (0, \mathsf{t}]. \tag{3.35}
$$

Secondly, we test (3.24) by ϑ_t. Being $\mathscr{L}_\mu$ monotone, we easily see that

$$
\int_\Omega \partial_t \mathscr{L}_\mu(\vartheta)\vartheta_t \ge 0 \qquad \text{a.e. in } (0, T)\,. \tag{3.36}
$$

We now integrate in time: taking into account (3.36), using Hölder's inequality and the Sobolev embeddings (2.4)–(2.5), we easily conclude

$$
\begin{aligned}
\varepsilon\|\vartheta_t\|^2_{L^2(0,t;V)} + \frac{1}{2}\|\nabla\vartheta(t)\|^2_H &\le \frac{1}{2}\|\nabla\vartheta^0\|^2_H + C\int_0^t \|\mathcal{K}\|_{L^4(\Gamma_c)}\|\vartheta\|_{L^4(\Gamma_c)}\|\vartheta_t\|_{L^2(\Gamma_c)} \\
&\quad + C\int_0^t \|\mathcal{K}\|_{L^6(\Gamma_c)}\|\widehat{\vartheta}_s\|_{L^3(\Gamma_c)}\|\vartheta_t\|_{L^2(\Gamma_c)} \\
&\quad + C\int_0^t \|\,\mathrm{div}(\overline{\mathbf{u}}_t)\|_H\,\|\vartheta_t\|_H + \int_0^t \|h\|_{V'}\|\vartheta_t\|_V \\
&\le \frac{1}{2}\|\vartheta_0\|^2_V + \frac{1}{2}\varepsilon\|\vartheta_t\|^2_{L^2(0,t;V)} \\
&\quad + C'_\varepsilon \int_0^t \|\mathcal{K}\|^2_{H^1(\Gamma_c)}\left(\|\vartheta\|^2_{L^4(\Gamma_c)} + \|\widehat{\vartheta}_s\|^2_{L^3(\Gamma_c)}\right) \\
&\quad + C_\varepsilon{}''\left(\int_0^t \|\overline{\mathbf{u}}_t\|^2_{\mathbf{W}} + \int_0^t \|h\|^2_{V'}\right).
\end{aligned}
\tag{3.37}
$$

Eventually, in view of (3.19), (3.18), (3.26), and (3.35), from (3.37) and the Poincaré inequality we deduce

$$
\|\vartheta_t\|_{L^2(0,t;V)} + \|\vartheta(t)\|_V \le M_2^\varepsilon \qquad \forall\, t \in (0, \mathsf{t}], \tag{3.38}
$$

from which (3.23) follows. □

Due to Lemma 3.5, we are in the position of defining the solution operator associated with (3.24)

$$
\mathcal{T}_2 : \pi_2(\mathcal{S}_\mathsf{t}) \times \mathcal{V}_\mathsf{t} \to \mathcal{W}_\mathsf{t} := \left\{\vartheta \in H^1(0,\mathsf{t};V) \ :\ \|\vartheta\|_{L^2(0,\mathsf{t};V)\cap L^\infty(0,\mathsf{t};L^1(\Omega))} \le M_2\right\}. \tag{3.39}
$$

Step 3. We fix $(\overline{\vartheta}, \widehat{\vartheta}_s, \overline{\chi}) \in \mathcal{W}_{\mathsf{t}} \times \pi_2(\mathcal{S}_{\mathsf{t}}) \times \pi_2(\mathcal{V}_{\mathsf{t}})$ and consider (the Cauchy problem) for (3.14) with data $(\overline{\vartheta}, \widehat{\vartheta}_s, \overline{\chi})$.

Lemma 3.6. *Assume* (2.H1), (2.H2), (2.H6), (2.H7), *and* (3.8).
Then, there exist M_3, $M_3^\varepsilon > 0$ *such that for all* $(\overline{\vartheta}, \widehat{\vartheta}_s, \overline{\chi}) \in \mathcal{W}_{\mathsf{t}} \times \pi_2(\mathcal{S}_{\mathsf{t}}) \times \pi_2(\mathcal{V}_{\mathsf{t}})$ *there exists a unique* $\vartheta_s \in H^1(0, \mathsf{t}; H^1(\Gamma_c))$, *with*

$$\|\vartheta_s\|_{L^2(0,\mathsf{t};H^1(\Gamma_c))\cap L^\infty(0,\mathsf{t};L^1(\Gamma_c))} \le M_3, \quad \|\partial_t \vartheta_s\|_{L^2(0,\mathsf{t};H^1(\Gamma_c))} \le M_3^\varepsilon\,, \tag{3.40}$$

such that ϑ_s *complies with the initial condition* (3.16), *and*

$$\varepsilon \int_{\Gamma_c} \partial_t \vartheta_s\, v + \int_{\Gamma_c} \partial_t \mathscr{L}_\mu(\vartheta_s) v - \int_{\Gamma_c} \partial_t \lambda(\overline{\chi})\, v + \int_{\Gamma_c} \nabla \vartheta_s\, \nabla v \tag{3.41}$$

$$+ \varepsilon \int_{\Gamma_c} \nabla \partial_t \vartheta_s\, \nabla v = \int_{\Gamma_c} k(\overline{\chi})(\overline{\vartheta} - \widehat{\vartheta}_s) v \quad \forall\, v \in H^1(\Gamma_c) \quad \textit{a.e. in } (0, \mathsf{t})\,. \tag{3.42}$$

Proof. Thanks to [17, Thms. 1,4] there exists a unique solution $\vartheta_s \in H^1(0, \mathsf{t}; H^1(\Gamma_c))$ to the Cauchy problem for (3.41). Hence, we shall just prove (3.40), referring to notation (3.25) for the term $k(\overline{\chi})$. We proceed as for (3.23): hence, we test (3.41) by ϑ_s and integrate in time. Developing the very same calculations as throughout (3.29)–(3.31), exploiting (2.H2) (via Lemma A.1), and recalling (3.8) and (3.9)–(3.10), we find

$$\frac{\varepsilon}{2}\|\vartheta_s(t)\|_{H^1(\Gamma_c)}^2 + C_1 \|\vartheta_s(t)\|_{L^1(\Gamma_c)} + \int_0^t \|\nabla \vartheta_s\|_{L^2(\Gamma_c)}^2 \le C + I_4 + I_5 + I_6\,,$$

where, also in view (2.H7),

$$I_4 = \int_0^t \int_{\Gamma_c} |\overline{\chi}_t| |\lambda'(\overline{\chi})| |\vartheta_s| \le C \int_0^t \|\overline{\chi}_t\|_{L^2(\Gamma_c)} \left(\|\overline{\chi}\|_{L^4(\Gamma_c)} + 1\right) \|\vartheta_s\|_{L^4(\Gamma_c)}\,, \tag{3.43}$$

and, in view of (3.26),

$$I_5 \le \int_0^t \|\mathcal{K}\|_{L^4(\Gamma_c)} \|\overline{\vartheta}\|_{L^4(\Gamma_c)} \|\vartheta_s\|_{L^2(\Gamma_c)} \le \|\mathcal{K}\|_{L^\infty(0,t;L^4(\Gamma_c))} \int_0^t \|\overline{\vartheta}\|_{L^4(\Gamma_c)} \|\vartheta_s\|_{L^2(\Gamma_c)}\,, \tag{3.44}$$

$$\begin{aligned} I_6 &= \int_0^t \int_{\Gamma_c} \|\mathcal{K}\|_{L^6(\Gamma_c)} \|\widehat{\vartheta}_s\|_{L^3(\Gamma_c)} \|\vartheta_s\|_{L^2(\Gamma_c)} \\ &\le \|\mathcal{K}\|_{L^\infty(0,t;L^6(\Gamma_c))} \int_0^t \|\widehat{\vartheta}_s\|_{L^3(\Gamma_c)} \|\vartheta_s\|_{L^2(\Gamma_c)}\,. \end{aligned} \tag{3.45}$$

Taking into account (3.18), (3.22), and (3.39), and estimating the term $\|\vartheta_s\|_{L^2(\Gamma_c)}$ by $\|\vartheta_s\|_{L^1(\Gamma_c)}$ and $\|\nabla \vartheta_s\|_{L^2(\Gamma_c)}^2$ in the same as in (3.32)–(3.33), we finally apply the Gronwall Lemma to conclude

$$\varepsilon^{1/2} \|\vartheta_s\|_{L^\infty(0,t;H^1(\Gamma_c))} + \|\vartheta_s\|_{L^2(0,t;H^1(\Gamma_c))\cap L^\infty(0,t;L^1(\Gamma_c))} \le M_3. \tag{3.46}$$

Then, we test (3.41) by $\partial_t \vartheta_s$ and integrate in time. Thanks to (3.46) and arguing in the same way as in (3.43)–(3.44), we find

$$\begin{aligned}\varepsilon \int_0^t \|\partial_t \vartheta_s\|_{H^1(\Gamma_c)}^2 + \frac{1}{2}\|\vartheta_s(t)\|_{H^1(\Gamma_c)}^2 + \int_0^t \int_{\Gamma_c} \partial_t \mathscr{L}_\mu(\vartheta_s)\partial_t \vartheta_s \\ \leq \frac{1}{2}\|\vartheta_s^0\|_{H^1(\Gamma_c)}^2 + \int_0^t \|\overline{\chi}_t\|_{L^2(\Gamma_c)} \|\overline{\chi}\|_{L^4(\Gamma_c)} \|\partial_t \vartheta_s\|_{L^4(\Gamma_c)} \\ + \int_0^t \|\mathcal{K}\|_{L^4(\Gamma_c)} \|\overline{\vartheta}\|_{L^4(\Gamma_c)} \|\partial_t \vartheta_s\|_{L^2(\Gamma_c)} \\ + \int_0^t \|\mathcal{K}\|_{L^6(\Gamma_c)} \|\widehat{\vartheta}_s\|_{L^3(\Gamma_c)} \|\partial_t \vartheta_s\|_{L^2(\Gamma_c)}\,.\end{aligned}$$

Again taking into account (3.18), (3.22), and (3.39), as well as the fact that the third summand on the left-hand side is nonnegative, we readily deduce the second part of (3.40). □

Lemma 3.6 enables us to define a solution operator associated with (3.41)

$$\mathcal{T}_3 : \mathcal{W}_\mathsf{t} \times \pi_2(\mathcal{S}_\mathsf{t}) \times \pi_2(\mathcal{V}_\mathsf{t}) \to \mathcal{Y}_\mathsf{t} := \left\{ \vartheta_s \in L^\infty(0,\mathsf{t};H^1(\Gamma_c)) \ : \ \|\vartheta_s\|_{L^2(0,\mathsf{t};H^1(\Gamma_c)) \cap L^\infty(0,\mathsf{t};L^1(\Gamma_c))} \leq M_3 \right\}.$$

3.3 Local existence for Problem $(\mathbf{P}_\varepsilon^\mu)$

Proposition 3.7. *Assume* (2.H1)–(2.H10), (2.16)–(2.17) *and* (3.7)–(3.10). *Then, there exists* $\widehat{T} \in (0,T]$, *possibly depending on* $\mu > 0$, *such that for every* $\varepsilon > 0$ *Problem* $(\mathbf{P}_\varepsilon^\mu)$ *admits a solution* $(\vartheta, \vartheta_s, \mathbf{u}, \chi, \xi, \boldsymbol{\eta})$ *on the interval* $(0, \widehat{T})$.

Proof. In view of the above Lemmata 3.4, 3.5, 3.6 we are able to define an operator $\mathcal{T}$ whose fixed points are solutions of Problem $(\mathbf{P}_\varepsilon^\mu)$.

Definition of $\mathcal{T}$. In the end, we define

$$\mathcal{T} : \mathcal{S}_\mathsf{t} \to \mathcal{W}_\mathsf{t} \times \mathcal{Y}_\mathsf{t} \times \pi_2(\mathcal{V}_\mathsf{t})$$

by setting for every $(\widehat{\vartheta}, \widehat{\vartheta}_s, \widehat{\chi}) \in \mathcal{S}_\mathsf{t}$

$$\mathcal{T}(\widehat{\vartheta}, \widehat{\vartheta}_s, \widehat{\chi}) := (\vartheta, \vartheta_s, \chi), \ \text{ where } \begin{cases} \vartheta = \mathcal{T}_2(\widehat{\vartheta}_s\,, \mathcal{T}_1(\widehat{\vartheta}, \widehat{\vartheta}_s, \widehat{\chi}))\,, \\ \vartheta_s = \mathcal{T}_3(\mathcal{T}_2(\widehat{\vartheta}_s,\, \mathcal{T}_1(\widehat{\vartheta}, \widehat{\vartheta}_s, \widehat{\chi})),\, \widehat{\vartheta}_s,\, \pi_2(\mathcal{T}_1(\widehat{\vartheta}, \widehat{\vartheta}_s, \widehat{\chi})))\,, \\ \chi = \pi_2(\mathcal{T}_1(\widehat{\vartheta}, \widehat{\vartheta}_s, \widehat{\chi}))\,. \end{cases} \tag{3.47}$$

Thus, in order to prove Proposition 3.7 it is sufficient to show that there exists $\widehat{T} \in (0,T]$ such that for every $\varepsilon > 0$

$$\mathcal{T} \text{ maps } \mathcal{S}_{\widehat{T}} \text{ into itself,} \tag{3.48}$$

$$\begin{gathered}\mathcal{T} : \mathcal{S}_{\widehat{T}} \to \mathcal{S}_{\widehat{T}} \quad \text{is compact and continuous} \\ \text{w.r.t. the topology of } L^2(0,\widehat{T};H) \times L^{5/2}(0,\widehat{T};L^3(\Gamma_c)) \times L^{10}(0,\widehat{T};L^6(\Gamma_c)).\end{gathered} \tag{3.49}$$

Ad (3.48)**.** We fix $(\widehat{\vartheta}, \widehat{\vartheta}_s, \widehat{\chi}) \in \mathcal{S}_\mathsf{t}$ and let $(\vartheta, \vartheta_s, \chi) = \mathcal{T}(\widehat{\vartheta}, \widehat{\vartheta}_s, \widehat{\chi})$. First of all, the three-dimensional version of the Gagliardo-Nirenberg inequality (cf. [23, p. 125]) yields

$$\|v\|_H \leq C\|\vartheta\|_V^{3/5}\|\vartheta\|_{L^1(\Omega)}^{2/5}, \tag{3.50}$$

so that, using (3.23), we get

$$\|\vartheta\|_{L^2(0,\mathsf{t};H)} \leq \|\vartheta\|^{3/5}_{L^2(0,\mathsf{t};V)} \|\vartheta\|^{2/5}_{L^2(0,\mathsf{t};L^1(\Omega))} \leq \mathsf{t}^{1/5} M_2 \,. \tag{3.51}$$

The Gagliardo-Nirenberg inequality in $2D$ gives

$$\|\vartheta_s\|_{L^3(\Gamma_c)} \leq C\|\vartheta_s\|^{2/3}_{H^1(\Gamma_c)}\|\vartheta_s\|^{1/3}_{L^1(\Gamma_c)},$$

whence

$$\|\vartheta_s\|_{L^{5/2}(0,\mathsf{t};L^3(\Gamma_c))} \leq \|\vartheta\|^{2/3}_{L^2(0,\mathsf{t};H^1(\Gamma_c))} \|\vartheta\|^{1/3}_{L^5(0,\mathsf{t};L^1(\Gamma_c))} \leq \mathsf{t}^{1/15} M_3 \,. \tag{3.52}$$

Finally, by (3.19) we have

$$\|\chi\|_{L^{10}(0,\mathsf{t});L^6(\Gamma_c)} \leq C\mathsf{t}^{1/10} M_1 \,, \tag{3.53}$$

the constant C depending on the Sobolev embedding (2.5). Clearly, there exists $\widehat{T}$ (which does not depend on ε) such that $(\vartheta, \vartheta_s, \chi)$ belongs to $\mathcal{S}_{\widehat{T}}$, hence the operator $\mathcal{T}$ maps $\mathcal{S}_{\widehat{T}}$ into itself.

Ad (3.49)**.** Exploiting (3.19), (3.23), and (3.40), the Sobolev embeddings (2.4)–(2.5) and [28, Thm. 4, Cor. 5], one sees immediately that the operator $\mathcal{T} : \mathcal{S}_{\widehat{T}} \to \mathcal{S}_{\widehat{T}}$ is compact. We shall prove that $\mathcal{T}$ is continuous in three steps, basically checking that the operators $\mathcal{T}_i$, $i = 1, 2, 3$ defined in Section 3.2 are continuous w.r.t. to suitable topologies.

We fix a sequence $\{(\widehat{\vartheta}_n, \widehat{\vartheta}_{s,n}, \widehat{\chi}_n)\} \subset \mathcal{S}_{\widehat{T}}$ such that there exists $(\widehat{\vartheta}_\infty, \widehat{\vartheta}_{s,\infty}, \widehat{\chi}_\infty) \in \mathcal{S}_{\widehat{T}}$ with

$$\widehat{\vartheta}_n \to \widehat{\vartheta}_\infty \quad \text{in } L^2(0, \widehat{T}; H) \quad \text{as } n \to \infty \,, \tag{3.54}$$

$$\widehat{\vartheta}_{s,n} \to \widehat{\vartheta}_{s,\infty} \quad \text{in } L^{5/2}(0, \widehat{T}; L^3(\Gamma_c)) \quad \text{as } n \to \infty \,, \tag{3.55}$$

$$\widehat{\chi}_n \to \widehat{\chi}_\infty \quad \text{in } L^{10}(0, \widehat{T}; L^6(\Gamma_c)) \quad \text{as } n \to \infty \,, \tag{3.56}$$

we let $(\mathbf{u}_n, \chi_n) := \mathcal{T}_1(\widehat{\vartheta}_n, \widehat{\vartheta}_{s,n}, \widehat{\chi}_n)$, and denote by $\{\boldsymbol{\eta}_n\}$ and ξ_n the associated sequences of selections of the graph α and β, respectively, such that (2.37) and (2.39) hold for all $n \in \mathbb{N}$. Due to (3.19), there exists a subsequence (which we do not relabel) and a quadruple

$$\begin{aligned}(\mathbf{u}_\infty, \boldsymbol{\eta}_\infty, \chi_\infty, \xi_\infty) \in H^1(0, \widehat{T}; \mathbf{W}) &\times L^2(0, \widehat{T}; H^{-1/2}(\Gamma_c)) \\ &\times (L^2(0, \widehat{T}; H^2(\Gamma_c)) \cap L^\infty(0, \widehat{T}; H^1(\Gamma_c)) \cap H^1(0, \widehat{T}; L^2(\Gamma_c))) \\ &\times L^2(0, \widehat{T}; L^2(\Gamma_c))\end{aligned}$$

such that the following convergences hold as $n \nearrow \infty$:

$$\begin{aligned}&\mathbf{u}_n \rightharpoonup \mathbf{u}_\infty \quad \text{in } H^1(0, \widehat{T}; \mathbf{W}), \qquad \mathbf{u}_n \to \mathbf{u}_\infty \quad \text{in } C^0([0, \widehat{T}]; (H^{1-s}(\Omega))^3) \ \text{ for all } s > 0, \\ &\mathbf{u}_n \to \mathbf{u}_\infty \quad \text{in } C^0([0, \widehat{T}]; (L^p(\Gamma_c))^3), \quad \text{for all } 1 \leq p < 4,\end{aligned} \tag{3.57}$$

$$\boldsymbol{\eta}_n \rightharpoonup \boldsymbol{\eta}_\infty \quad \text{in } L^2(0, \widehat{T}; (H^{-1/2}(\Gamma_c))^3) \,, \tag{3.58}$$

$$\begin{aligned}&\chi_n \rightharpoonup^* \chi_\infty \quad \text{in } H^1(0, \widehat{T}; L^2(\Gamma_c)) \cap L^\infty(0, \widehat{T}; H^1(\Gamma_c)) \cap L^2(0, \widehat{T}; H^2(\Gamma_c)), \\ &\chi_n \to \chi_\infty \quad \text{in } C^0([0, \widehat{T}]; H^{1-s}(\Gamma_c)) \cap L^2(0, \widehat{T}; H^{2-s}(\Gamma_c)) \ \text{ for all } s > 0 \,,\end{aligned} \tag{3.59}$$

$$\xi_n \rightharpoonup \xi_\infty \quad \text{in } L^2(0, \widehat{T}; L^2(\Gamma_c)) \,. \tag{3.60}$$

Thanks to these convergences, and recalling (2.H7), it is possible to pass to the limit in the PDE system (3.20) as $n \nearrow \infty$ (see the proof of [4, Prop. 4.7] for further details). Thus, we prove that the quadruple $(\mathbf{u}_\infty, \boldsymbol{\eta}_\infty, \chi_\infty, \xi_\infty)$ fulfils (3.20), with data $(\widehat{\vartheta}_\infty, \widehat{\vartheta}_{s,\infty}, \widehat{\chi}_\infty)$, on $(0, \widehat{T})$.

Furthermore, the pair $(\mathbf{u}_\infty, \chi_\infty)$ clearly complies with the initial conditions (2.31)–(2.30). Note that the identifications

$$\xi_\infty \in \beta(\chi_\infty), \quad \boldsymbol{\eta}_\infty \in \alpha(\mathbf{u}_\infty)$$

are proved by semicontinuity arguments (see [2, Lemma 1.3, p 42]). In the end, recalling the definition of the operator $\mathcal{T}_1$ we conclude that

$$(\mathbf{u}_\infty, \chi_\infty) = \mathcal{T}_1(\widehat{\vartheta}_\infty, \widehat{\vartheta}_{s,\infty}, \widehat{\chi}_\infty)\,. \tag{3.61}$$

In fact, by uniqueness of the limit we readily deduce that convergences (3.57)–(3.60) hold for the whole sequences. In particular, we have checked that (3.54)–(3.56) imply

$$\mathcal{T}_1(\widehat{\vartheta}_n, \widehat{\vartheta}_{s,n}, \widehat{\chi}_n) \to \mathcal{T}_1(\widehat{\vartheta}_\infty, \widehat{\vartheta}_{s,\infty}, \widehat{\chi}_\infty) \qquad \text{in the sense of (3.57), (3.59).} \tag{3.62}$$

We now consider the sequence $\vartheta_n := \mathcal{T}_2(\widehat{\vartheta}_{s,n}, \mathbf{u}_n, \chi_n) = \mathcal{T}_2(\widehat{\vartheta}_{s,n}, \mathcal{T}_1(\widehat{\vartheta}_n, \widehat{\vartheta}_{s,n}, \widehat{\chi}_n))$ for all $n \in \mathbb{N}$. Thanks to (3.23), $\{\vartheta_n\}$ is bounded in $H^1(0,\widehat{T},V)$, hence there exists $\vartheta_\infty \in H^1(0,\widehat{T},V)$ such that (up to a subsequence)

$$\begin{aligned} &\vartheta_n \rightharpoonup \vartheta_\infty \quad \text{in} \;\; H^1(0,\widehat{T};V), \qquad \vartheta_n \to \vartheta_\infty \quad \text{in} \;\; C^0([0,\widehat{T}];H^{1-s}(\Omega)) \;\; \forall\, s>0, \\ &\vartheta_n \to \vartheta_\infty \;\; \text{in} \;\; C^0([0,\widehat{T}];L^p(\Gamma_c)) \quad \text{for all } 1 \le p < 4, \end{aligned} \tag{3.63}$$

as $n \nearrow \infty$. In particular, ϑ_∞ complies with (3.15). Moreover, using that $\mathscr{L}_\mu$ is Lipschitz continuous, taking into account the strong convergence for ϑ_n specified by the second of (3.63) and recalling that, by maximal monotonicity, the graph of $\mathscr{L}_\mu$ is strongly-weakly closed, it is not difficult to conclude that

$$\begin{cases} \mathscr{L}_\mu(\vartheta_n) \rightharpoonup^* \mathscr{L}_\mu(\vartheta_\infty) & \text{in } L^\infty(0,\widehat{T};V) \cap H^1(0,\widehat{T};H), \\ \mathscr{L}_\mu(\vartheta_n) \to \mathscr{L}_\mu(\vartheta_\infty) & \text{in } C^0([0,\widehat{T}];H). \end{cases} \tag{3.64}$$

Furthermore, it follows from (3.55), (3.59), (3.63), and the Lipschitz continuity of k that, among others, the following convergences hold $n \nearrow \infty$:

$$\begin{aligned} k(\chi_n)\vartheta_n \to k(\chi_\infty)\vartheta_\infty \quad &\text{in } L^\infty(0,\widehat{T};L^2(\Gamma_c)) \text{ for all } 1 \le p < 4, \text{ and} \\ k(\chi_n)\widehat{\vartheta}_{s,n} \to k(\chi_\infty)\widehat{\vartheta}_{s,\infty} \quad &\text{in } L^2(0,\widehat{T};L^2(\Gamma_c))\,. \end{aligned} \tag{3.65}$$

Combining (3.57) with (3.63)–(3.65), we pass to the limit as $n \nearrow \infty$ in (3.24) with data $(\widehat{\vartheta}_{s,n}, \mathbf{u}_n, \chi_n)$, and we deduce that ϑ_∞ solves (the Cauchy problem) for equation (3.24), with the triple $(\widehat{\vartheta}_{s,\infty}, \mathbf{u}_\infty, \chi_\infty)$, on $(0,\widehat{T})$. Thus,

$$\vartheta_\infty = \mathcal{T}_2(\widehat{\vartheta}_{s,\infty}, \mathbf{u}_\infty, \chi_\infty) = \mathcal{T}_2(\widehat{\vartheta}_{s,\infty}, \mathcal{T}_1(\widehat{\vartheta}_\infty, \widehat{\vartheta}_{s,\infty}, \widehat{\chi}_\infty))\,,$$

the second equality ensuing from (3.61). Again, since the limit ϑ_∞ does not depend on the subsequence in (3.63), it turns out the convergences specified therein hold along the whole sequence $\{\vartheta_n\}$. In conclusion,

$$\mathcal{T}_2(\widehat{\vartheta}_{s,n}, \mathcal{T}_1(\widehat{\vartheta}_n, \widehat{\vartheta}_{s,n}, \widehat{\chi}_n)) \to \mathcal{T}_2(\widehat{\vartheta}_{s,\infty}, \mathcal{T}_1(\widehat{\vartheta}_\infty, \widehat{\vartheta}_{s,\infty}, \widehat{\chi}_\infty)) \qquad \text{in the sense of (3.63).} \tag{3.66}$$

Finally, we let

$$\vartheta_{s,n} := \mathcal{T}_3(\vartheta_n, \widehat{\vartheta}_{s,n}, \chi_n) = \mathcal{T}_3(\mathcal{T}_2(\widehat{\vartheta}_{s,n},\, \mathcal{T}_1(\widehat{\vartheta}_n, \widehat{\vartheta}_{s,n}, \widehat{\chi}_n)),\, \widehat{\vartheta}_{s,n},\, \pi_2(\mathcal{T}_1(\widehat{\vartheta}_n, \widehat{\vartheta}_{s,n}, \widehat{\chi}_n)))$$

for every $n \in \mathbb{N}$. By (3.40), $\{\vartheta_{s,n}\}$ is bounded in $H^1(0,\widehat{T};H^1(\Gamma_c))$. Thus, there exists a (not relabeled) subsequence and $\vartheta_{s,\infty} \in H^1(0,\widehat{T};H^1(\Gamma_c))$ such that

$$\begin{aligned} &\vartheta_{s,n} \rightharpoonup \vartheta_{s,\infty} && \text{in } H^1(0,\widehat{T};H^1(\Gamma_c)), \\ &\vartheta_{s,n} \to \vartheta_{s,\infty} && \text{in } C^0([0,\widehat{T}];H^{1-s}(\Gamma_c)) \text{ for all } s>0, \end{aligned} \tag{3.67}$$

$$\begin{aligned} &\mathscr{L}_\mu(\vartheta_{s,n}) \rightharpoonup^* \mathscr{L}_\mu(\vartheta_{s,\infty}) && \text{in } L^\infty(0,\widehat{T};H^1(\Gamma_c)) \cap H^1(0,\widehat{T};L^2(\Gamma_c)), \\ &\mathscr{L}_\mu(\vartheta_{s,n}) \to \mathscr{L}_\mu(\vartheta_{s,\infty}) && \text{in } C^0([0,\widehat{T}];L^2(\Gamma_c)), \end{aligned} \tag{3.68}$$

where the latter convergences can be proved arguing in the very same way as for (3.64). In particular, $\vartheta_{s,\infty}$ fulfils initial condition (3.16), and, like in the proof of Lemma 3.6, combining convergences (3.67)–(3.68) we conclude that $\vartheta_{s,\infty}$ fulfils (3.41) on $(0,\widehat{T})$, with data $(\vartheta_\infty, \widehat{\vartheta}_{s,\infty}, \chi_\infty)$. Now, (3.59) and the growth properties of λ (see (2.H7)) clearly yield that

$$\lambda'(\chi_n)\partial_t\chi_n \rightharpoonup \lambda'(\chi_\infty)\partial_t\chi_\infty \quad \text{in } L^2(0,\widehat{T};H^1(\Gamma_c)'). \tag{3.69}$$

Then, exploiting (3.59) and (3.67) we easily check that

$$k(\chi_n)\vartheta_{s,n} \to k(\chi_\infty)\vartheta_{s,\infty} \qquad \text{in } L^\infty(0,\widehat{T};L^2(\Gamma_c)). \tag{3.70}$$

Collecting (3.67)–(3.70) and also taking into account (3.65), we pass to the limit in (3.41) (with data $(\vartheta_n, \widehat{\vartheta}_{s,n}, \chi_n)$) as $n \nearrow \infty$. Hence,

$$\vartheta_{s,\infty} = \mathcal{T}_3(\vartheta_\infty, \widehat{\vartheta}_{s,\infty}, \chi_\infty) = \mathcal{T}_3(\mathcal{T}_2(\widehat{\vartheta}_{s,\infty},\, \mathcal{T}_1(\widehat{\vartheta}_\infty, \widehat{\vartheta}_{s,\infty}, \widehat{\chi}_\infty)),\, \widehat{\vartheta}_{s,\infty},\, \pi_2(\mathcal{T}_1(\widehat{\vartheta}_\infty, \widehat{\vartheta}_{s,\infty}, \widehat{\chi}_\infty))),$$

and we again deduce that convergences (3.67)–(3.68) hold for the whole sequences $\{\vartheta_{s,n}\}$. Eventually, we have that

$$\begin{aligned} &\mathcal{T}_3(\mathcal{T}_2(\widehat{\vartheta}_{s,n},\, \mathcal{T}_1(\widehat{\vartheta}_n, \widehat{\vartheta}_{s,n}, \widehat{\chi}_n)),\, \widehat{\vartheta}_{s,n},\, \pi_2(\mathcal{T}_1(\widehat{\vartheta}_n, \widehat{\vartheta}_{s,n}, \widehat{\chi}_n))) \to \\ &\mathcal{T}_3(\mathcal{T}_2(\widehat{\vartheta}_{s,\infty},\, \mathcal{T}_1(\widehat{\vartheta}_\infty, \widehat{\vartheta}_{s,\infty}, \widehat{\chi}_\infty)),\, \widehat{\vartheta}_{s,\infty},\, \pi_2(\mathcal{T}_1(\widehat{\vartheta}_\infty, \widehat{\vartheta}_{s,\infty}, \widehat{\chi}_\infty))) \quad \text{in the sense of (3.67).} \end{aligned} \tag{3.71}$$

Clearly, (3.62), (3.66) and (3.71) show that $\mathcal{T}$ is continuous in the sense of (3.49). □

3.4 Global existence for Problem $(\mathbf{P}_\varepsilon^\mu)$

In order to show that the local solution to Problem $(\mathbf{P}_\varepsilon^\mu)$ actually extends to the whole time interval $(0,T)$, we shall obtain some global in time estimates on the solution components $(\vartheta, \vartheta_s, \mathbf{u}, \chi)$ and then use a fairly standard argument to conclude that, for every $\varepsilon > 0$ and $\mu > 0$, the local solution found in Proposition 3.7 extends to the (unique, by the calculations in Section 3.5) global solution of Problem $(\mathbf{P}_\varepsilon^\mu)$.

Lemma 3.8 (Global estimates)**.** *Assume* (2.H1)–(2.H10) *and let* $(\vartheta^0, \vartheta_s^0, \mathbf{u}_0, \chi_0)$ *be a quadruple of initial data complying with conditions* (3.7)–(3.10) *and* (2.16)–(2.17)*. Then, for every* $\mu > 0$ *there exists a constant* $C_\mu > 0$*, depending on the problem data, on* M_0 *(cf.* (3.9)*) and* M_0^μ *(cf.* (3.10)*), but neither on* $\mathsf{t} \in (0,T]$ *nor on* $\varepsilon > 0$*, such that for every solution* $(\vartheta, \vartheta_s, \mathbf{u}, \chi, \xi, \boldsymbol{\eta})$ *to Problem* $(\mathbf{P}_\varepsilon^\mu)$ *on the interval* $(0,\mathsf{t})$ *there holds*

$$\varepsilon^{1/2}\|\vartheta\|_{L^\infty(0,\mathsf{t};V)} + \|\vartheta\|_{L^2(0,\mathsf{t};V)\cap L^\infty(0,\mathsf{t};L^1(\Omega))} + \|j_\mu^*(\mathscr{L}_\mu(\vartheta))\|_{L^\infty(0,\mathsf{t};L^1(\Omega))} \le C_\mu\,, \tag{3.72}$$

$$\begin{aligned} &\varepsilon^{1/2}\|\vartheta_s\|_{L^\infty(0,\mathsf{t};H^1(\Gamma_c))} + \|\vartheta_s\|_{L^2(0,\mathsf{t};H^1(\Gamma_c))\cap L^\infty(0,\mathsf{t};L^1(\Gamma_c))} \\ &\qquad\qquad + \|j_\mu^*(\mathscr{L}_\mu(\vartheta_s))\|_{L^\infty(0,\mathsf{t};L^1(\Gamma_c))} \le C_\mu\,, \end{aligned} \tag{3.73}$$

$$\|\chi\|_{H^1(0,\mathsf{t};L^2(\Gamma_c))\cap L^\infty(0,\mathsf{t};H^1(\Gamma_c))} \le C_\mu\,, \tag{3.74}$$

$$\|\mathbf{u}\|_{H^1(0,\mathsf{t};\mathbf{W})} \le C_\mu. \tag{3.75}$$

Proof. We test (2.36) by $\mathbf{u}_t$: owing to (2.7)–(2.8) and to (2.H3), via the chain rule for $\widehat{\alpha}$ and the Hölder inequality we obtain

$$\begin{aligned}&\frac{C_b}{2}\int_0^t \|\mathbf{u}_t\|^2_{\mathbf{W}} + \frac{C_a}{2}\|\mathbf{u}(t)\|^2_{\mathbf{W}} + \int_\Omega \vartheta\,\mathrm{div}(\mathbf{u}_t) + \widehat{\alpha}(\mathbf{u}(t)) + \int_0^t\int_{\Gamma_c} \chi\,\mathbf{u}\cdot\mathbf{u}_t \\ &\leq C\|\mathbf{u}_0\|^2_W + \widehat{\alpha}(\mathbf{u}_0) + \frac{1}{2C_b}\int_0^T \|\mathbf{F}\|^2_{V'}\,.\end{aligned} \tag{3.76}$$

Next, we multiply (2.38) by χ_t and integrate in time. Again applying the chain rule to the functional $\widehat{\beta}$, and using (2.H5), with easy calculations we find

$$\begin{aligned}&\frac{1}{2}\int_0^t \|\chi_t\|^2_{L^2(\Gamma_c)} + \frac{1}{2}\|\nabla\chi(t)\|^2_{L^2(\Gamma_c)} + \int_{\Gamma_c}\widehat{\beta}(\chi(t)) \\ &\qquad \leq C + \frac{1}{2}\|\nabla\chi_0\|^2_{L^2(\Gamma_c)} + \int_{\Gamma_c}\widehat{\beta}(\chi_0) - \int_0^t\int_{\Gamma_c}\lambda'(\chi)\chi_t\vartheta_s + I_7 + I_8\,,\end{aligned} \tag{3.77}$$

where

$$I_7 = L_\sigma^2\int_0^t\int_{\Gamma_c}|\chi|^2 \leq 2L_\sigma^2 T\|\chi_0\|_{L^2(\Gamma_c)} + 2L_\sigma^2 T\int_0^t\left(\int_0^s \|\chi_t\|^2_{L^2(\Gamma_c)}\right)\mathrm{d}s \tag{3.78}$$

while, with an easy integration by parts, we have

$$I_8 = -\frac{1}{2}\int_0^t\int_{\Gamma_c}\chi_t|\mathbf{u}|^2 = \int_0^t\int_{\Gamma_c}\chi\mathbf{u}_t\mathbf{u} - \frac{1}{2}\int_{\Gamma_c}\chi(t)|\mathbf{u}(t)|^2 + \frac{1}{2}\int_{\Gamma_c}\chi_0|\mathbf{u}_0|^2\,. \tag{3.79}$$

Furthermore, being $\widehat{\beta}$ convex, we have

$$\begin{aligned}\int_{\Gamma_c}\widehat{\beta}(\chi(t)) &\geq -C_{1,\beta}\|\chi(t)\|_{L^1(\Gamma_c)} - C_{2,\beta} \\ &\geq -\eta\|\chi(t)\|^2_{L^2(\Gamma_c)} - C_\eta \\ &\geq -2\eta T\int_0^t \|\chi_t\|^2_{L^2(\Gamma_c)} - 2\eta\|\chi_0\|^2_{L^2(\Gamma_c)} - C_\eta\end{aligned} \tag{3.80}$$

for some suitable $\eta > 0$ to be specified later. Finally, we test (3.13) by ϑ, (3.14) by ϑ_s, integrate in time and develop in both cases the same computations as throughout (3.29)–(3.31). We add the resulting inequalities with (3.76) and (3.77): also taking into account (3.79), some terms cancel out. Furthermore, choosing $\eta \leq 1/8T$ in (3.80) and applying the Gronwall Lemma to deal with the integral term on the right-hand side of (3.78), we arrive at

$$\begin{aligned}&\varepsilon\|\vartheta(t)\|^2_V + \|\vartheta\|^2_{L^2(0,t;V)} + \|\vartheta(t)\|_{L^1(\Omega)} + \varepsilon\|\vartheta_s(t)\|^2_{H^1(\Gamma_c)} + \|\vartheta_s\|^2_{L^2(0,t;H^1(\Gamma_c))} \\ &+ \|\vartheta_s(t)\|_{L^1(\Gamma_c)} + \int_0^t\int_{\Gamma_c}k(\chi)(\vartheta-\vartheta_s)^2 + \|\mathbf{u}_t\|^2_{L^2(0,t;\mathbf{W})} + \|\mathbf{u}(t)\|^2_{\mathbf{W}} \\ &+ \int_{\Gamma_c}\chi(t)|\mathbf{u}(t)|^2 + \widehat{\alpha}(\mathbf{u}(t)) + \|\chi_t\|^2_{L^2(0,t;L^2(\Gamma_c))} + \|\nabla\chi(t)\|^2_{L^2(\Gamma_c)} \\ &\leq C\left(1 + \int_0^t \|h\|^2_{V'} + \int_0^t \|\mathbf{F}\|^2_{\mathbf{W}'}\right).\end{aligned} \tag{3.81}$$

for every $t \in [0,T]$. Noting that the tenth and eleventh summands on the right-hand side are nonnegative due to (2.H3) and (2.H4), we conclude. Ultimately, we also find that

$$\|j^*_\mu(\mathscr{L}_\mu(\vartheta))\|_{L^\infty(0,\mathrm{t};L^1(\Omega))} + \|j^*_\mu(\mathscr{L}_\mu(\vartheta_s))\|_{L^\infty(0,\mathrm{t};L^1(\Gamma_c))} \leq C\,, \tag{3.82}$$

and (3.72)–(3.75) ensue. □

Remark 3.9. As it is clear from the above proof, constant C_μ in (3.72)–(3.75) depends on μ only through M_0^μ (3.10), i.e. the bound for $\varepsilon^{1/2}\|\vartheta^0_{\varepsilon\mu}\|_V$ and $\varepsilon^{1/2}\|\vartheta^0_{s,\varepsilon\mu}\|_{H^1(\Gamma_c)}$.

3.5 Uniqueness for Problem $(\mathbf{P}_\varepsilon^\mu)$.

We prove here the uniqueness statement in Theorem 3.1. Let us consider two families of solutions $(\vartheta_i, \vartheta_{s,i}, \mathbf{u}_i, \chi_i, \xi_i, \boldsymbol{\eta}_i)$, $i = 1, 2$, to Problem $(\mathbf{P}_\varepsilon^\mu)$. Hereafter, we shall refer to the notation

$$\widetilde{\vartheta} = \vartheta_1 - \vartheta_2, \quad \widetilde{\vartheta}_s = \vartheta_{s,1} - \vartheta_{s,2}, \quad \widetilde{\mathbf{u}} = \mathbf{u}_1 - \mathbf{u}_2, \quad \widetilde{\chi} = \chi_1 - \chi_2 \quad \widetilde{\xi} = \xi_1 - \xi_2, \quad \widetilde{\boldsymbol{\eta}} = \boldsymbol{\eta}_1 - \boldsymbol{\eta}_2.$$

We will derive suitable contracting estimates on the solutions. First, we subtract (3.13) written for $(\vartheta_2, \vartheta_{s,2}, \mathbf{u}_2, \chi_2)$ from (3.13) written for $(\vartheta_1, \vartheta_{s,1}, \mathbf{u}_1, \chi_1)$ and integrate in time. We get

$$\begin{aligned} &\varepsilon \int_\Omega \widetilde{\vartheta}(t)\, v + \int_\Omega (\mathscr{L}_\mu(\vartheta_1(t)) - \mathscr{L}_\mu(\vartheta_2(t)))\, v + \varepsilon \int_\Omega \nabla\widetilde{\vartheta}(t)\,\nabla v + \int_\Omega (1 * \nabla\widetilde{\vartheta})(t)\,\nabla v \\ &= \int_\Omega \operatorname{div} \widetilde{\mathbf{u}}(t)\, v + \int_{\Gamma_c} \Big(1 * [k(\chi_2)(\vartheta_2 - \vartheta_{s,2}) - k(\chi_1)(\vartheta_1 - \vartheta_{s,1})]\Big)(t)\, v \end{aligned} \tag{3.83}$$

for all $v \in V$. Letting $v = \widetilde{\vartheta}$, integrating in time and exploiting the monotonicity of $\mathscr{L}_\mu$, we obtain

$$\varepsilon \int_0^t \|\widetilde{\vartheta}\|_V^2 + \frac{1}{2}\|(1 * \nabla\widetilde{\vartheta})(t)\|_H^2 \le \int_0^t \int_\Omega \operatorname{div} \widetilde{\mathbf{u}}\, \widetilde{\vartheta} + \sum_{j=9}^{11} I_j. \tag{3.84}$$

In order to estimate the three latter summands, we proceed as follows. Using Hölder's and Young's inequality (cf. also (2.1)), and well-known Sobolev embeddings, we find

$$\begin{aligned} I_9 &= \left| \int_0^t \int_{\Gamma_c} \Big(1 * \big[(k(\chi_2) - k(\chi_1))(\vartheta_1 - \vartheta_{s,1})\big]\Big)\widetilde{\vartheta} \right| \\ &\le \int_0^t \|\widetilde{\vartheta}\|_{L^2(\Gamma_c)} \left\|1 * \big[(k(\chi_2) - k(\chi_1))(\vartheta_1 - \vartheta_{s,1})\big]\right\|_{L^2(\Gamma_c)} \\ &\le C \int_0^t \|\widetilde{\vartheta}(s)\|_{L^2(\Gamma_c)} \|(k(\chi_2) - k(\chi_1))(\vartheta_1 - \vartheta_{s,1})\|_{L^2(0,s;L^2(\Gamma_c))}\, \mathrm{d}s \\ &\le C \int_0^t \|\widetilde{\vartheta}(s)\|_{L^2(\Gamma_c)} \|k(\chi_2) - k(\chi_1)\|_{L^2(0,s;L^4(\Gamma_c))} \|\vartheta_1 - \vartheta_{s,1}\|_{L^\infty(0,s;L^4(\Gamma_c))}\, \mathrm{d}s \\ &\le \delta \int_0^t \|\widetilde{\vartheta}\|_V^2 + c_{\delta,1} \int_0^t \|\widetilde{\chi}\|^2_{L^2(0,s;H^1(\Gamma_c))}\, \mathrm{d}s\,, \end{aligned} \tag{3.85}$$

for a suitable positive δ to be chosen later. In particular, the positive constant $c_{\delta,1}$ also depends on the a priori estimate (3.72) on $\|\vartheta_1 - \vartheta_{s,1}\|_{L^\infty(0,T;L^4(\Gamma_c))}$. Arguing similarly, also taking into account (3.74) in order to estimate χ_2, we have

$$\begin{aligned} I_{10} &= \left| \int_0^t \int_{\Gamma_c} \Big(1 * \big[k(\chi_2)\widetilde{\vartheta}\big]\Big)\widetilde{\vartheta} \right| \le \int_0^t \|\widetilde{\vartheta}\|_{L^2(\Gamma_c)} \left\|1 * \big[k(\chi_2)\widetilde{\vartheta}\big]\right\|_{L^2(\Gamma_c)} \\ &\le C \int_0^t \|\widetilde{\vartheta}(s)\|_{L^2(\Gamma_c)} \|\chi_2 + 1\|_{L^\infty(0,s;L^4(\Gamma_c))} \|\widetilde{\vartheta}\|_{L^2(0,s;L^4(\Gamma_c))}\, \mathrm{d}s \\ &\le \delta \int_0^t \|\widetilde{\vartheta}\|_V^2 + c_{\delta,2} \int_0^t \|\widetilde{\vartheta}\|^2_{L^2(0,s;V)}\, \mathrm{d}s\,, \end{aligned} \tag{3.86}$$

and

$$
\begin{aligned}
I_{11} &= \left| \int_0^t \int_{\Gamma_c} \Big(1 * \big[k(\chi_2)\widetilde{\vartheta}_s\big]\Big)\widetilde{\vartheta} \right| \leq \int_0^t \|\widetilde{\vartheta}\|_{L^2(\Gamma_c)} \, \|1 * \big[k(\chi_2)\widetilde{\vartheta}_s\big]\|_{L^2(\Gamma_c)} \\
&\leq C \int_0^t \|\widetilde{\vartheta}(s)\|_{L^2(\Gamma_c)} \, \|\chi_2 + 1\|_{L^\infty(0,s;L^4(\Gamma_c))} \|\widetilde{\vartheta}_s\|_{L^2(0,s;L^4(\Gamma_c))} \, \mathrm{d}s \\
&\leq \delta \int_0^t \|\widetilde{\vartheta}\|_V^2 + c_{\delta,3} \int_0^t \|\widetilde{\vartheta}_s\|^2_{L^2(0,s;H^1(\Gamma_c))} \, \mathrm{d}s \,.
\end{aligned} \tag{3.87}
$$

In a similar way, we consider (3.14) written for $(\vartheta_2, \vartheta_{s,2}, \chi_2)$ and for $(\vartheta_1, \vartheta_{s,1}, \chi_1)$, we take the difference, test the resulting equation by $\widetilde{\vartheta}_s$ and we integrate in time. We obtain

$$
\varepsilon \int_0^t \|\widetilde{\vartheta}_s\|^2_{H^1(\Gamma_c)} + \frac{1}{2} \|(1 * \nabla \widetilde{\vartheta}_s)(t)\|^2_{L^2(\Gamma_c)} \leq I_{12} + I_{13} \,, \tag{3.88}
$$

where the terms I_{12} and I_{13} are estimated in the following way. Recalling (2.11), we find

$$
\begin{aligned}
I_{12} &= \int_0^t \int_{\Gamma_c} \Big(\lambda(\chi_1) - \lambda(\chi_2)\Big) \widetilde{\vartheta}_s \\
&\leq C \int_0^t \|\widetilde{\chi}\|_{L^2(\Gamma_c)} \big(\|\chi_1\|_{L^4(\Gamma_c)} + \|\chi_2\|_{L^4(\Gamma_c)} + 1\Big) \|\widetilde{\vartheta}_s\|_{L^4(\Gamma_c)} \\
&\leq \delta \int_0^t \|\widetilde{\vartheta}_s\|^2_{H^1(\Gamma_c))} + c_{\delta,4} \int_0^t \|\widetilde{\chi}\|^2_{L^2(\Gamma_c)} \, \mathrm{d}s \,.
\end{aligned} \tag{3.89}
$$

where the constant $\delta > 0$ is the same as in (3.85)–(3.87). Note that the positive constant $c_{\delta,4}$ also depends on the estimate for $\|\chi_1\|_{L^\infty(0,T;L^4(\Gamma_c))}$ and $\|\chi_2\|_{L^\infty(0,T;L^4(\Gamma_c))}$ given by (3.74). Moreover, arguing as in the derivation of (3.85)–(3.87), we have

$$
\begin{aligned}
I_{13} &= \left| \int_0^t \int_{\Gamma_c} \Big(1 * \big[k(\chi_1)(\vartheta_1 - \vartheta_{s,1}) - k(\chi_2)(\vartheta_2 - \vartheta_{s,2})\big]\Big)\widetilde{\vartheta}_s \right| \\
&\leq \delta \int_0^t \|\widetilde{\vartheta}_s\|^2_{L^2(\Gamma_c)} + c_{\delta,5} \int_0^t \|\widetilde{\chi}\|^2_{L^2(0,s;H^1(\Gamma_c))} \\
&+ c_{\delta,6} \int_0^t \|\widetilde{\vartheta}\|^2_{L^2(0,s;V)} + c_{\delta,7} \int_0^t \|\widetilde{\vartheta}_s\|^2_{L^2(0,s;H^1(\Gamma_c))} \, \mathrm{d}s \,.
\end{aligned} \tag{3.90}
$$

Now, we subtract (2.36), written for $(\vartheta_2, \mathbf{u}_2, \chi_2, \boldsymbol{\eta}_2)$, from (2.36), written for $(\vartheta_1, \mathbf{u}_1, \chi_1, \boldsymbol{\eta}_1)$, we test the resulting relation by $\widetilde{\mathbf{u}}$ and integrate on $(0, t)$. Recalling (2.7)–(2.8), and the monotonicity of α (cf. (2.H3)), we end up with

$$
\begin{aligned}
&\frac{C_b}{2} \|\widetilde{\mathbf{u}}(t)\|^2_W + C_a \|\widetilde{\mathbf{u}}\|^2_{L^2(0,t;W)} + \int_0^t \int_{\Gamma_c} \operatorname{div} \widetilde{\vartheta}\, \widetilde{\mathbf{u}} \\
&\qquad \leq - \int_0^t \!\! \int_{\Gamma_c} \chi_2 \, (\widetilde{\mathbf{u}})^2 - \int_0^t \!\! \int_{\Gamma_c} \widetilde{\chi}\, \mathbf{u}_1 \, \widetilde{\mathbf{u}} \leq \int_0^t \|\widetilde{\chi}\|_{L^2(\Gamma_c)} \|\mathbf{u}_1\|_{L^4(\Gamma_c)} \|\widetilde{\mathbf{u}}\|_{L^4(\Gamma_c)} \\
&\qquad \leq C \int_0^t \|\widetilde{\chi}\|^2_{L^2(\Gamma_c)} + C \int_0^t \|\widetilde{\mathbf{u}}\|^2_W \,,
\end{aligned} \tag{3.91}
$$

where the last two inequalities follow from Hölder's and Young's inequalities, from Sobolev's embeddings and from the fact that $\chi_2 \geq 0$ a.e. on $(0, T) \times \Gamma_c$, due to (2.H4). In particular, the constant C in (3.91) depends on $\|\mathbf{u}_1\|_{L^\infty(0,T;L^4(\Gamma_c))}$ through estimate (3.75).

On the other hand, let us consider the difference of (2.38) written for $(\vartheta_{s,1}, \mathbf{u}_1, \chi_1, \xi_1)$ and (2.38) for $(\vartheta_{s,2}, \mathbf{u}_2, \chi_2, \xi_2)$, multiply it by $\widetilde{\chi}$ and integrate on $(0,t) \times \Gamma_c$. Taking (2.H4), (2.H5), and (2.H7) into account, we get

$$\begin{aligned}
&\frac{1}{2}\|\widetilde{\chi}(t)\|^2_{L^2(\Gamma_c)} + \int_0^t \|\nabla\widetilde{\chi}\|^2_{L^2(\Gamma_c)} \leq \int_0^t\int_{\Gamma_c} (\sigma'(\chi_2) - \sigma'(\chi_1))\widetilde{\chi} - \int_0^t\int_{\Gamma_c} \lambda'(\chi_1)\,\widetilde{\vartheta}_s\,\widetilde{\chi} \\
&- \int_0^t\int_{\Gamma_c} \big(\lambda'(\chi_1) - \lambda'(\chi_2)\big)\,\vartheta_{s,2}\widetilde{\chi} - \frac{1}{2}\int_0^t\int_{\Gamma_c} (\mathbf{u}_1 + \mathbf{u}_2)\cdot\widetilde{\mathbf{u}}\widetilde{\chi} \\
&\leq L_\sigma \int_0^t \|\widetilde{\chi}\|^2_{L^2(\Gamma_c)} + C\int_0^t \big(\|\chi_1\|_{L^4(\Gamma_c)} + 1\big)\|\widetilde{\vartheta}_s\|_{L^4(\Gamma_c)}\,\|\widetilde{\chi}\|_{L^2(\Gamma_c)} \\
&+ C\int_0^t \|\vartheta_{s,2}\|_{L^4(\Gamma_c)}\,\|\widetilde{\chi}\|_{L^4(\Gamma_c)}\,\|\widetilde{\chi}\|_{L^2(\Gamma_c)} + C\int_0^t \|\mathbf{u}_1 + \mathbf{u}_2\|_{L^4(\Gamma_c)}\,\|\widetilde{\mathbf{u}}\|_{L^4(\Gamma_c)}\,\|\widetilde{\chi}\|_{L^2(\Gamma_c)} \\
&\leq \delta\int_0^t \|\widetilde{\vartheta}_s\|^2_{H^1(\Gamma_c)} + \delta\int_0^t \|\widetilde{\chi}\|^2_{H^1(\Gamma_c)} + C\int_0^t \|\widetilde{\mathbf{u}}\|^2_{\mathbf{W}} + c_{\delta,8}\int_0^t \|\widetilde{\chi}\|^2_{L^2(\Gamma_c)}\,,
\end{aligned} \tag{3.92}$$

where the constant $c_{\delta,8}$ depends on estimates (3.73)–(3.75) for the quantities $\|\chi_1\|_{L^\infty(0,T;L^4(\Gamma_c))}$, $\|\vartheta_{s,2}\|_{L^\infty(0,T;L^4(\Gamma_c))}$, and on $\|\mathbf{u}_1 + \mathbf{u}_2\|_{L^\infty(0,T;L^4(\Gamma_c))}$.

Finally, we add (3.84), (3.88), (3.91), and (3.92). Noting that two terms cancel out and taking into account (3.85)–(3.87) and (3.89)–(3.90) (in which we choose, e.g., $\delta = \min\{\varepsilon/6, 1/2\}$), we find

$$\begin{aligned}
\varepsilon\int_0^t \|\widetilde{\vartheta}\|^2_V + \varepsilon\int_0^t \|\widetilde{\vartheta}_s\|^2_{H^1(\Gamma_c)} &+ \int_0^t \|\widetilde{\chi}\|^2_{H^1(\Gamma_c)} + \|\widetilde{\chi}(t)\|^2_{L^2(\Gamma_c)} + C_b\|\widetilde{\mathbf{u}}(t)\|^2_{\mathbf{W}} \\
&\leq C\Big(\int_0^t \|\widetilde{\vartheta}\|^2_{L^2(0,s;V)} + \int_0^t \|\widetilde{\vartheta}_s\|^2_{L^2(0,s;H^1(\Gamma_c))} \\
&+ \int_0^t \|\widetilde{\chi}\|^2_{L^2(0,s;H^1(\Gamma_c))}\,\mathrm{d}s + \int_0^t \|\widetilde{\chi}\|^2_{L^2(\Gamma_c)} + \int_0^t \|\widetilde{\mathbf{u}}\|^2_{\mathbf{W}}\Big)\,.
\end{aligned}$$

Thus, by Gronwall's Lemma, we conclude that $\vartheta_1 = \vartheta_2$, $\vartheta_{s,1} = \vartheta_{s,2}$, $\mathbf{u}_1 = \mathbf{u}_2$, and $\chi_1 = \chi_2$. A comparison in (2.36) and (2.38) also yields $\boldsymbol{\eta}_1 = \boldsymbol{\eta}_2$ and $\xi_1 = \xi_2$, so that the uniqueness statement in Theorem 3.1 follows. □

4 Proof of Theorem 1

As we mentioned in Remark 3.2, we shall pass to the limit in Problem $(\mathbf{P}^\mu_\varepsilon)$ first as $\varepsilon \searrow 0$ and $\mu > 0$ is fixed (cf. with Proposition 4.4), and then as $\mu \searrow 0$ (see Section 4.2). The next result (whose proof is postponed to the Appendix) concerns the construction of sequences of initial data $\{\vartheta^0_{\varepsilon\mu}\} \subset V$ and $\{\vartheta^0_{s,\varepsilon\mu}\} \subset H^1(\Gamma_c)$ for Problem $(\mathbf{P}^\mu_\varepsilon)$ complying with (3.7)–(3.10), and such that the sequence of solutions to Problem $(\mathbf{P}^\mu_\varepsilon)$, supplemented with the data $(\vartheta^0_{\varepsilon\mu}, \vartheta^0_{s,\varepsilon\mu}, \mathbf{u}_0, \chi_0)$, converges to a solution of Problem $(\mathbf{P})$ in the two consecutive limit procedures $\varepsilon \searrow 0$ and $\mu \searrow 0$. As, in our construction, the data in fact depend only on the parameter $\mu > 0$, we shall denote them as ϑ^0_μ and $\vartheta^0_{s,\mu}$ for simplicity.

Lemma 4.1. *Assume that the initial data w_0 and z_0 respectively comply with* (2.14) *and* (2.15). *Then,*

1. *there exists a sequence $\{w^0_\mu\}_\mu \subset V$ fulfilling for every $\mu > 0$*

$$\|w^0_\mu\|_H \leq \|w_0\|_H\,, \tag{4.1}$$

$$\int_\Omega j^*\left(w^0_\mu\right) \leq \int_\Omega j^*\left(w_0\right)\,, \tag{4.2}$$

and such that

$$w_\mu^0 \to w_0 \qquad \textit{in } H \quad \textit{as } \mu \searrow 0. \tag{4.3}$$

Furthermore, let us set

$$\vartheta_\mu^0 := \mathscr{L}_\mu^{-1}(w_\mu^0) \quad \textit{for all } \mu > 0\,. \tag{4.4}$$

There exists a constant $C_{w_0}^\mu > 0$, depending on w_0 and on $\mu > 0$ but independent of $\varepsilon > 0$, such that for all $\varepsilon > 0$

$$\varepsilon^{1/2}\|\vartheta_\mu^0\|_V \le C_{w_0}^\mu\,. \tag{4.5}$$

2. *There exists a sequence $\{z_\mu^0\}_\mu \subset H^1(\Gamma_c)$ fulfilling for every $\mu > 0$*

$$\|z_\mu^0\|_{L^2(\Gamma_c)} \le \|z_0\|_{L^2(\Gamma_c)}\,, \tag{4.6}$$

$$\int_{\Gamma_c} j^*\left(z_\mu^0\right) \le \int_{\Gamma_c} j^*\left(z_0\right)\,, \tag{4.7}$$

and such that

$$z_\mu^0 \to z_0 \qquad \textit{in } L^2(\Gamma_c) \quad \textit{as } \mu \searrow 0. \tag{4.8}$$

Furthermore, setting

$$\vartheta_{s,\mu}^0 := \mathscr{L}_\mu^{-1}(z_\mu^0) \quad \textit{for all } \mu > 0\,, \tag{4.9}$$

there exists a constant $C_{z_0}^\mu > 0$, depending on z_0 and on $\mu > 0$ but independent of $\varepsilon > 0$, such that for all $\varepsilon > 0$

$$\varepsilon^{1/2}\|\vartheta_{s,\mu}^0\|_{H^1(\Gamma_c)} \le C_{z_0}^\mu\,. \tag{4.10}$$

Remark 4.2. Note that our construction of the sequences $\{\vartheta_\mu^0\}$ and $\{\vartheta_{s,\mu}^0\}$ only depends on the data w_0 and z_0. This is, ultimately, the main reason why we have chosen to approximate the operator ℓ by the regularization $\mathscr{L}_\mu$, instead of the usual Yosida regularization ℓ_μ. Indeed, if we had used the latter approximation of ℓ, starting from the datum w_0 we should have constructed approximate data $\vartheta_\mu^0 \in V$ satisfying the corresponding bound

$$\|(j_\mu)^*(\ell_\mu(\vartheta_\mu^0))\|_{L^1(\Omega)} \le C \qquad \text{for all } \mu > 0, \tag{4.11}$$

(where $(j_\mu)^*$ is the conjugate of the Yosida approximation j_μ of j), cf. with the proof of Lemma 3.5 (clearly, the same considerations hold for $\vartheta_{s,\mu}^0$). To deduce (4.11) from the condition $j^*(w_0) \in L^1(\Omega)$, one should virtually choose ϑ_μ^0 in such a way that $\ell_\mu(\vartheta_\mu^0) = w_0$ (leaving aside the condition $\vartheta_\mu^0 \in V$). However, it is not clear to us how to carry out this construction, since ℓ_μ is not invertible. Instead, $\mathscr{L}_\mu$ can be inverted, and the calculations we shall provide in the proof of Lemma 4.1 show that the sequence defined by (4.4) complies with (3.9)–(3.10).

Notation 4.3. We shall denote by

$$\{(\vartheta_{\varepsilon\mu}, \vartheta_{s,\varepsilon\mu}, \mathbf{u}_{\varepsilon\mu}, \chi_{\varepsilon\mu}, \xi_{\varepsilon\mu}, \boldsymbol{\eta}_{\varepsilon\mu})\} \text{ the sequence of solutions to Problem } (\mathbf{P}_\varepsilon^\mu)$$
$$\text{with initial data } \{(\vartheta_\mu^0, \vartheta_{s,\mu}^0, \mathbf{u}_0, \chi_0)\}.$$

Further, for simplicity we shall use the notation

$$w_{\varepsilon\mu} := \mathscr{L}_\mu(\vartheta_{\varepsilon\mu}), \qquad z_{\varepsilon\mu} := \mathscr{L}_\mu(\vartheta_{s,\varepsilon\mu})\,,$$

so that, in view of (4.4) and of (4.9) respectively, we have

$$w_{\varepsilon\mu}(0) = w_\mu^0, \qquad z_{\varepsilon\mu}(0) = z_\mu^0 \qquad \text{for all } \varepsilon > 0. \tag{4.12}$$

4.1 Passage to the limit in $(\mathbf{P}_\varepsilon^\mu)$ as $\varepsilon \searrow 0$

Proposition 4.4. *Let $\mu > 0$ be fixed. Under the assumptions of Theorem 1, there exists a (not relabeled) subsequence and $(\vartheta_\mu, w_\mu, \vartheta_{s,\mu}, z_\mu, \mathbf{u}_\mu, \chi_\mu, \xi_\mu, \boldsymbol{\eta}_\mu)$ such that the following convergences hold as $\varepsilon \searrow 0$*

$$\vartheta_{\varepsilon\mu} \rightharpoonup \vartheta_\mu \quad \text{in } L^2(0,T;V), \qquad \varepsilon\mathcal{R}(\vartheta_{\varepsilon\mu}) \to 0 \quad \text{in } L^\infty(0,T;V'), \tag{4.13}$$

$$\vartheta_{s,\varepsilon\mu} \rightharpoonup \vartheta_{s,\mu} \quad \text{in } L^2(0,T;H^1(\Gamma_c)), \qquad \varepsilon\mathcal{R}_{\Gamma_c}(\vartheta_{s,\varepsilon\mu}) \to 0 \quad \text{in } L^\infty(0,T;H^1(\Gamma_c)'), \tag{4.14}$$

$$w_{\varepsilon\mu} \rightharpoonup w_\mu \quad \text{in } L^2(0,T;V)\,, \tag{4.15}$$

$$\varepsilon\mathcal{R}(\vartheta_{\varepsilon\mu}) + w_{\varepsilon\mu} \rightharpoonup w_\mu \quad \text{in } H^1(0,T;V')\,, \tag{4.16}$$

$$z_{\varepsilon\mu} \rightharpoonup z_\mu \quad \text{in } L^2(0,T;H^1(\Gamma_c))\,, \tag{4.17}$$

$$\varepsilon\mathcal{R}_{\Gamma_c}(\vartheta_{s,\varepsilon\mu}) + z_{\varepsilon\mu} \rightharpoonup z_\mu \quad \text{in } H^1(0,T;H^1(\Gamma_c)')\,, \tag{4.18}$$

$$\begin{aligned} &\chi_{\varepsilon\mu} \rightharpoonup^* \chi_\mu \quad \text{in } H^1(0,T;L^2(\Gamma_c)) \cap L^\infty(0,T;H^1(\Gamma_c)) \cap L^2(0,T;H^2(\Gamma_c)),\\ &\chi_{\varepsilon\mu} \to \chi_\mu \quad \text{in } C^0([0,T];H^{1-\delta}(\Gamma_c)) \cap L^2(0,T;H^{2-\delta}(\Gamma_c)) \quad \text{for all } \delta > 0, \end{aligned} \tag{4.19}$$

$$\xi_{\varepsilon\mu} \rightharpoonup \xi_\mu \quad \text{in } L^2(0,T;L^2(\Gamma_c)), \tag{4.20}$$

$$\boldsymbol{\eta}_{\varepsilon\mu} \rightharpoonup \boldsymbol{\eta}_\mu \quad \text{in } L^2(0,T;\mathbf{W}'), \tag{4.21}$$

$$\begin{aligned} &\mathbf{u}_{\varepsilon\mu} \rightharpoonup \mathbf{u}_\mu \quad \text{in } H^1(0,T;\mathbf{W}),\\ &\mathbf{u}_{\varepsilon\mu} \to \mathbf{u}_\mu \quad \text{in } C^0([0,T];H^{1-\delta}(\Omega)^3) \quad \text{for all } \delta > 0\,. \end{aligned} \tag{4.22}$$

Moreover, w_μ and z_μ have the further regularity

$$\begin{aligned} &w_\mu \in L^\infty(0,T;H), \qquad j_\mu^*(w_\mu) \in L^\infty(0,T;L^1(\Omega))\,,\\ &z_\mu \in L^\infty(0,T;L^2(\Gamma_c)), \qquad j_\mu^*(z_\mu) \in L^\infty(0,T;L^1(\Gamma_c))\,, \end{aligned} \tag{4.23}$$

and satisfy

$$\begin{cases} w_\mu(x,t) = \mathscr{L}_\mu(\vartheta_\mu(x,t)) \quad \text{for a.e.} \, (x,t) \in \Omega \times (0,T),\\ z_\mu(x,t) = \mathscr{L}_\mu(\vartheta_{s,\mu}(x,t)) \quad \text{for a.e.} \, (x,t) \in \Gamma_c \times (0,T)\,. \end{cases} \tag{4.24}$$

Further, the functions $(\vartheta_\mu, w_\mu, \vartheta_{s,\mu}, z_\mu, \mathbf{u}_\mu, \chi_\mu, \xi_\mu, \boldsymbol{\eta}_\mu)$ fulfil equations (2.36)–(2.40) *and* (3.13)–(3.14) *with $\varepsilon = 0$, and the quadruple $(w_\mu, z_\mu, \mathbf{u}_\mu, \chi_\mu)$ complies with the initial conditions*

$$\begin{aligned} &w_\mu(0) = w_\mu^0 \quad \text{a.e. in}\, \Omega, \quad z_\mu(0) = z_\mu^0 \quad \text{a.e. in } \Gamma_c,\\ &\mathbf{u}_\mu(0) = \mathbf{u}_0 \quad \text{a.e. in } \Omega, \quad \chi_\mu(0) = \chi_0 \quad \text{a.e. in } \Gamma_c. \end{aligned} \tag{4.25}$$

Notation 4.5. Hereafter, we shall call $(\mathbf{P}^\mu)$ the boundary value problem given by (2.36)–(2.40) and (3.13)–(3.14) with $\varepsilon = 0$, supplemented with relations (4.24) and the initial conditions (4.25).

Proof. Let us point out that, thanks to (4.1)–(4.5) and (4.6)–(4.7), all estimates in Lemma 3.8 hold for the sequence $\{(\vartheta_{\varepsilon\mu}, w_{\varepsilon\mu}, \vartheta_{s,\varepsilon\mu}, z_{\varepsilon\mu}, \mathbf{u}_{\varepsilon\mu}, \chi_{\varepsilon\mu}, \xi_{\varepsilon\mu}, \boldsymbol{\eta}_{\varepsilon\mu})\}_\varepsilon$ with a constant C_μ only depending on the problem data, on the initial data $(w_0, z_0, \mathbf{u}_0, \chi_0)$, and on $\mu > 0$, but independent of $\varepsilon > 0$. In order to pass to the limit in $(\mathbf{P}_\varepsilon^\mu)$ as $\varepsilon \searrow 0$, we need some further estimates in addition to (3.72)–(3.75). Using the latter bounds and arguing by comparison in (3.13) and in (3.14), we conclude that there exists $C > 0$ such that for all ε, $\mu > 0$

$$\|\varepsilon\mathcal{R}(\partial_t\vartheta_{\varepsilon\mu}) + \partial_t w_{\varepsilon\mu}\|_{L^2(0,T;V')} + \|\varepsilon\mathcal{R}_{\Gamma_c}(\partial_t\vartheta_{s,\varepsilon\mu}) + \partial_t z_{\varepsilon\mu}\|_{L^2(0,T;H^1(\Gamma_c)')} \le C\,. \tag{4.26}$$

Similarly, a comparison in (2.36) leads to

$$\|\boldsymbol{\eta}_{\varepsilon\mu}\|_{L^2(0,T;\mathbf{W}')} \le C \quad \text{for all } \varepsilon,\, \mu > 0. \tag{4.27}$$

Finally, we test (2.38) by $\xi_{\varepsilon\mu} \in \beta(\chi_{\varepsilon\mu})$ and get by standard arguments

$$\|\xi_{\varepsilon\mu}\|_{L^2(0,T;L^2(\Gamma_c))} + \|\chi_{\varepsilon\mu}\|_{L^2(0,T;H^2(\Gamma_c))} \leq C \quad \text{for all } \varepsilon,\, \mu > 0. \tag{4.28}$$

Moreover, in view of the Lipschitz continuity of $\mathscr{L}_\mu$ (cf. with (3.6)), joint with estimates (3.72) and (3.73) for ϑ and ϑ_s, we conclude that there exists a constant $\overline{C_\mu} > 0$, depending on the problem data and on $\mu > 0$ but independent of $\varepsilon > 0$, such that

$$\|w_{\varepsilon\mu}\|_{L^2(0,T;V)} + \|z_{\varepsilon\mu}\|_{L^2(0,T;H^1(\Gamma_c))} \leq \overline{C_\mu} \quad \text{for all } \varepsilon > 0. \tag{4.29}$$

Combining estimates (3.72)–(3.75) and (4.27)–(4.29) with the Ascoli-Arzelà theorem, the well-known [28, Them. 4, Cor. 5], and standard weak compactness results, we find that there exists an eight-uple $(\vartheta_\mu, w_\mu, \vartheta_{s,\mu}, z_\mu, \mathbf{u}_\mu, \chi_\mu, \boldsymbol{\eta}_\mu, \xi_\mu)$ such that, along a suitable (not relabeled) subsequence, convergences (4.13)–(4.15), (4.17), and (4.19)–(4.22) hold. Clearly, (4.16) follows from (4.26) and the second of (4.13). In the same way, we obtain (4.18). Therefore, the further regularity (4.23) for w_μ and z_μ ensues from the continuous embeddings $L^2(0,T;V) \cap H^1(0,T;V') \subset C^0([0,T];H)$ and $L^2(0,T;H^1(\Gamma_c)) \cap H^1(0,T;H^1(\Gamma_c)') \subset C^0([0,T];L^2(\Gamma_c))$. In order to prove the second of (4.23), we exploit a *Lebesgue point* argument. Indeed, by (4.15) and the lower-semicontinuity of the integral functional induced by (the convex function) j_μ^* with respect to the weak convergence in $L^2(0,T;V)$, we find that for all $t_0 \in (0,T)$ and $r > 0$ such that $(t_0 - r, t_0 + r) \subset (0,T)$

$$\int_{t_0-r}^{t_0+r} \int_\Omega j_\mu^*(w_\mu) \leq \liminf_{\varepsilon \searrow 0} \left(\int_{t_0-r}^{t_0+r} \int_\Omega j_\mu^*(w_{\varepsilon\mu}) \right) \leq 2r \sup_{\varepsilon>0} \|j_\mu^*(w_{\varepsilon\mu})\|_{L^\infty(0,T;L^1(\Omega))} \leq 2r\overline{C}\,, \tag{4.30}$$

the latter inequality due to (3.72). We now divide the above relation by r and let $r \downarrow 0$. Using that the Lebesgue point property holds at almost every $t_0 \in (0,T)$, we obtain the estimate

$$\|j_\mu^*(w_\mu)\|_{L^\infty(0,T;L^1(\Omega))} \leq \overline{C} \qquad \text{for all } \mu > 0, \tag{4.31}$$

$\overline{C}$ being the same constant as in (3.72). The analogous bound for $j_\mu^*(z_\mu)$ is proved in the same way.

Furthermore, we remark that (2.H6), (2.H7), (4.13)–(4.19), trace theorems, and Sobolev embeddings yield that, as $\varepsilon \searrow 0$,

$$\begin{aligned} &k(\chi_{\varepsilon\mu})(\vartheta_{\varepsilon\mu} - \vartheta_{s,\varepsilon\mu}) \rightharpoonup k(\chi_\mu)(\vartheta_\mu - \vartheta_{s,\mu}) \quad \text{in } L^2(0,T;L^2(\Gamma_c)),\\ &\big(1 * k(\chi_{\varepsilon\mu})(\vartheta_{\varepsilon\mu} - \vartheta_{s,\varepsilon\mu})\big) \to \big(1 * k(\chi_\mu)(\vartheta_\mu - \vartheta_{s,\mu})\big) \quad \text{in } C^0([0,T];H^{-1/2}(\Gamma_c)), \end{aligned} \tag{4.32}$$

as well as

$$\begin{aligned} &\lambda'(\chi_{\varepsilon\mu})\vartheta_{s,\varepsilon\mu} \rightharpoonup \lambda'(\chi_\mu)\vartheta_{s,\mu} \quad \text{in } L^2(0,T;L^2(\Gamma_c)),\\ &\lambda(\chi_{\varepsilon\mu}) \rightharpoonup \lambda(\chi_\mu) \quad \text{in } H^1(0,T;H^1(\Gamma_c)'). \end{aligned} \tag{4.33}$$

In order to pass to the limit in Problem $(\mathbf{P}_\varepsilon^\mu)$ as $\varepsilon \searrow 0$, we use the above convergences and proceed as in Section 3.3. In particular, for (2.36)–(2.40) we exploit (4.13), (4.19)–(4.22) and argue in the same way as in [4, Prop. 4.7], to which we refer the reader. Further, relying on (4.13)–(4.19) and the above (4.32)–(4.33), we also pass to the limit as $\varepsilon \searrow 0$ both in (3.13) and in (3.14). We thus conclude that $(\vartheta_\mu, w_\mu, \vartheta_{s,\mu}, z_\mu, \mathbf{u}_\mu, \chi_\mu, \boldsymbol{\eta}_\mu, \xi_\mu)$ satisfies the PDE system given by (2.36)–(2.40) (3.13)–(3.14) with $\varepsilon = 0$.

Finally, it remains to show (4.24). We shall just prove the relation for w_μ, the argument for z_μ being completely analogous. By maximal monotonicity of $\mathscr{L}_\mu$ (cf. [2, Lemma 1.3, p. 42]), it is sufficient to show that

$$\limsup_{\varepsilon \searrow 0} \int_0^T \int_\Omega w_{\varepsilon\mu}\vartheta_{\varepsilon\mu} \leq \int_0^T \int_\Omega w_\mu \vartheta_\mu\,. \tag{4.34}$$

which we prove integrating in time (3.13), testing it by $\vartheta_{\varepsilon\mu}$, and again integrating on $(0,T)$. Hence, we get

$$
\begin{aligned}
&\limsup_{\varepsilon\searrow 0}\int_0^T\int_\Omega w_{\varepsilon\mu}\vartheta_{\varepsilon\mu} \\
&\leq \limsup_{\varepsilon\searrow 0}\int_0^T\int_\Omega \left(\varepsilon\vartheta_\mu^0 + w_\mu^0 - \mathrm{div}(\mathbf{u}_0)\right)\vartheta_{\varepsilon\mu} + \limsup_{\varepsilon\searrow 0}\int_0^T\int_\Omega \varepsilon\nabla\vartheta_\mu^0\nabla\vartheta_{\varepsilon\mu} - \varepsilon\liminf_{\varepsilon\searrow 0}\int_0^T \|\vartheta_{\varepsilon\mu}\|_V^2 \\
&\quad + \limsup_{\varepsilon\searrow 0}\int_0^T\int_\Omega \mathrm{div}(\mathbf{u}_{\varepsilon\mu})\vartheta_{\varepsilon\mu} - \liminf_{\varepsilon\searrow 0}\int_\Omega |(1*\nabla\vartheta_{\varepsilon\mu})(T)|^2 \\
&\quad - \liminf_{\varepsilon\searrow 0}\int_0^T\int_{\Gamma_c}\left(1*k(\chi_{\varepsilon\mu})(\vartheta_{\varepsilon\mu}-\vartheta_{s,\varepsilon\mu})\right)\vartheta_{\varepsilon\mu} + \limsup_{\varepsilon\searrow 0}\int_0^T \langle 1*h,\vartheta_{\varepsilon\mu}\rangle \\
&\leq \int_0^T\int_\Omega w_\mu\vartheta_\mu\,,
\end{aligned}
$$

where the second passage follows from (4.5) and the first of (4.13), from the second of (4.13), from (2.32) and the following relations:

$$
\lim_{\varepsilon\searrow 0}\int_0^T \langle 1*h,\vartheta_{\varepsilon\mu}\rangle = \int_0^T \langle 1*h,\vartheta_\mu\rangle, \qquad \lim_{\varepsilon\searrow 0}\int_0^T\int_\Omega \mathrm{div}(\mathbf{u}_{\varepsilon\mu})\vartheta_{\varepsilon\mu} = \int_0^T\int_\Omega \mathrm{div}(\mathbf{u}_\mu)\vartheta_\mu
$$

(due to (4.13) and (4.22), yielding in particular that $\mathrm{div}(\mathbf{u}_{\varepsilon\mu}) \to \mathrm{div}(\mathbf{u}_\mu)$ in $L^\infty(0,T;L^{6/5}(\Omega))$ as $\varepsilon\searrow 0$),

$$
\liminf_{\varepsilon\searrow 0}\int_\Omega |(1*\nabla\vartheta_{\varepsilon\mu})(T)|^2 \geq \int_\Omega |(1*\nabla\vartheta_\mu)(T)|^2
$$

(by the lower semicontinuity of the norm, since $1*\nabla\vartheta_{\varepsilon\mu} \rightharpoonup^* 1*\nabla\vartheta_\mu$ in $L^\infty(0,T;H)$ as $\varepsilon\searrow 0$), and, finally,

$$
\lim_{\varepsilon\searrow 0}\int_0^T\int_{\Gamma_c}\left(1*k(\chi_{\varepsilon\mu})(\vartheta_{\varepsilon\mu}-\vartheta_{s,\varepsilon\mu})\right)\vartheta_{\varepsilon\mu} = \int_0^T\int_{\Gamma_c}\left(1*k(\chi_\mu)(\vartheta_\mu-\vartheta_{s,\mu})\right)\vartheta_\mu
$$

thanks to the second of (4.32) and (4.13) (which gives that $\vartheta_{\varepsilon\mu} \rightharpoonup \vartheta_\mu$ in $L^2(0,T;H^{1/2}(\Gamma_c))$.

In the end, we show (4.25). The initial conditions for $\mathbf{u}_\mu$ and χ_μ ensue from (4.3) and convergences (4.19) and (4.22). On the other hand, thanks to (4.5) there holds

$$
\varepsilon\mathcal{R}(\vartheta_\mu^0) \to 0 \qquad \text{as } \varepsilon\searrow 0 \text{ in } V',
$$

hence by (4.12) and (4.16),

$$
w_\mu^0 = \lim_{\varepsilon\searrow 0} w_{\varepsilon\mu}(0) = \lim_{\varepsilon\searrow 0}\left(\varepsilon\mathcal{R}(\vartheta_\mu^0) + w_{\varepsilon\mu}(0)\right) = w_\mu(0)\,,
$$

where all of the above limits are meant with respect, e.g., to the $H^2(\Omega)'$-topology (in fact, to the topology of any space $\mathcal{Z}$ such that $V' \subset \mathcal{Z}$ with compact embedding). With a completely analogous argument, we prove the initial condition for z_μ as well. □

4.2 Conclusion of the proof of Theorem 1.

We are now going to show that the sequence $\{(w_\mu, \vartheta_\mu, z_\mu, \vartheta_{s,\mu}, \mathbf{u}_\mu, \chi_\mu, \xi_\mu, \boldsymbol{\eta}_\mu)\}_\mu$ of solutions to Problem $(\mathbf{P}^\mu)$ (cf. with Notation 4.5) obtained in Proposition 4.4 admits a subsequence converging as $\mu \searrow 0$ to a solution of Problem $(\mathbf{P})$. To this aim, we point out that there exists a constant $\overline{C} > 0$, *independent of* $\mu > 0$, such that

$$\begin{cases} \|\vartheta_\mu\|_{L^2(0,T;V)\cap L^\infty(0,T;L^1(\Omega))} + \|j^*_\mu(\mathscr{L}_\mu(\vartheta_\mu))\|_{L^\infty(0,T;L^1(\Omega))} \leq \overline{C}\,, \\ \|\vartheta_{s,\mu}\|_{L^2(0,T;H^1(\Gamma_c))\cap L^\infty(0,T;L^1(\Gamma_c))} + \|j^*_\mu(\mathscr{L}_\mu(\vartheta_{s,\mu}))\|_{L^\infty(0,T;L^1(\Gamma_c))} \leq \overline{C}\,, \\ \|\chi_\mu\|_{H^1(0,T;L^2(\Gamma_c))\cap L^\infty(0,T;H^1(\Gamma_c))} \leq \overline{C}\,, \\ \|\mathbf{u}_\mu\|_{H^1(0,T;\mathbf{W})} \leq \overline{C}. \end{cases} \tag{4.35}$$

This can be proved by testing (3.13) (with $\varepsilon = 0$) by ϑ_μ, (3.14) (with $\varepsilon = 0$) by $\vartheta_{s,\mu}$, (2.36) by $\mathbf{u}_\mu$, and (2.38) by χ_μ, adding the resulting equations and integrating in time. Developing the very same calculations as for Lemma 3.8, we conclude (4.35), whence (4.28), as well, for a constant independent of $\mu > 0$. Likewise, arguing by comparison in the equations satisfied by ϑ_μ and $\vartheta_{s,\mu}$ one obtains that there exists $C > 0$ such that for all $\mu > 0$

$$\|\partial_t w_\mu\|_{L^2(0,T;V')} + \|\boldsymbol{\eta}_\mu\|_{L^2(0,T;\mathbf{W}')} + \|\partial_t z_\mu\|_{L^2(0,T;H^1(\Gamma_c)')} \leq C\,. \tag{4.36}$$

Further, we are in the position of proving the following crucial estimate

$$\|w_\mu\|_{L^\infty(0,T;H)} + \|z_\mu\|_{L^\infty(0,T;L^2(\Gamma_c))} \leq C \tag{4.37}$$

for a constant independent of $\mu > 0$. Indeed, let us test (3.13), with $\varepsilon = 0$, by $w_\mu = \mathscr{L}_\mu(\vartheta_\mu)$, (3.14), with $\varepsilon = 0$, by $z_\mu = \mathscr{L}_\mu(\vartheta_{s,\mu})$, add the resulting relations and integrate on some time interval $(0,t)$, $t \in (0,T]$ (note that these estimates may be performed rigorously since $w_\mu \in L^2(0,T;V)$ and $z_\mu \in L^2(0,T;H^1(\Gamma_c))$ for all $\mu > 0$). Easy calculations lead to

$$\begin{aligned} &\frac{1}{2}\|w_\mu(t)\|^2_H + \int_0^t\int_\Omega \nabla\vartheta_\mu \nabla w_\mu + \frac{1}{2}\|z_\mu(t)\|^2_{L^2(\Gamma_c)} + \int_0^t\int_{\Gamma_c} \nabla\vartheta_{s,\mu}\nabla z_\mu \\ &+ \int_0^t\int_{\Gamma_c} k(\chi_\mu)\,(\vartheta_\mu - \vartheta_{s,\mu})\,(w_\mu - z_\mu) \leq \frac{1}{2}\|w^0_\mu\|^2_H + \frac{1}{2}\|z^0_\mu\|^2_{L^2(\Gamma_c)} + I_{14} + I_{15} + I_{16}\,, \end{aligned} \tag{4.38}$$

where

$$I_{14} = \int_0^t\int_\Omega |\operatorname{div}(\partial_t \mathbf{u}_\mu)||w_\mu| \leq C\|\mathbf{u}_\mu\|^2_{H^1(0,T;\mathbf{W})} + \frac{1}{2}\int_0^t \|w_\mu\|^2_H\,, \tag{4.39}$$

$$I_{15} = \int_0^t\int_\Omega |h||w_\mu| \leq \int_0^t \|h\|_H \|w_\mu\|_H\,, \tag{4.40}$$

$$\begin{aligned} I_{16} = \int_0^t\int_{\Gamma_c} |\partial_t\lambda(\chi_\mu)||z_\mu| &\leq C\int_0^t \|\partial_t\chi_\mu\|_{L^2(\Gamma_c)}\left(\|\chi_\mu\|_{L^\infty(\Gamma_c)} + 1\right)\|z_\mu\|_{L^2(\Gamma_c)} \\ &\leq C\|\chi_\mu\|^2_{H^1(0,T;L^2(\Gamma_c))} + C'\int_0^t \left(\|\chi_\mu\|^2_{L^\infty(\Gamma_c)} + 1\right)\|z_\mu\|^2_{L^2(\Gamma_c)}\,, \end{aligned} \tag{4.41}$$

the first inequality in (4.41) ensuing from (2.12). Now, we remark that the second and the fourth term on the left-hand side of (4.38) are nonnegative thanks to (4.24) and the monotonicity of $\mathscr{L}_\mu$. Combining the fact that k takes positive values (cf. (2.H6)) with the latter monotonicity argument (indeed, it can be easily checked that the trace $w_\mu|_{\Gamma_c}$ of w_μ on Γ_c fulfils $w_\mu|_{\Gamma_c} =$

$\mathscr{L}_\mu(\vartheta_\mu|_{\Gamma_c})$), we conclude that the fifth term (in the l.h.s. of (4.38)) is nonnegative as well. Thus, we collect (4.38)–(4.41): recalling (2.H8), estimates (4.1) and (4.6) on the initial data w_μ^0 and z_μ^0, as well as estimates (4.35) and (4.28) (the latter yields a bound for χ_μ in $L^2(0,T;L^\infty(\Gamma_c))$), and applying the Gronwall lemma, we end up with (4.37) as desired.

All of the above estimates, the Ascoli-Arzelà theorem, [28, Them. 4, Cor. 5], and standard weak compactness results yield that there exist a subsequence of $\{(w_\mu, \vartheta_\mu, z_\mu, \vartheta_{s,\mu}, \mathbf{u}_\mu, \chi_\mu, \xi_\mu, \boldsymbol{\eta}_\mu)\}_\mu$ (which we do not relabel) and functions $(w, \vartheta, z, \vartheta_s, \mathbf{u}, \chi, \xi, \boldsymbol{\eta})$ for which the first of (4.13), the first of (4.14), convergences (4.19)–(4.22) and

$$
\begin{aligned}
& w_\mu \rightharpoonup^* w \quad \text{in } L^\infty(0,T;H) \cap H^1(0,T;V')\,, \\
& w_\mu \to w \quad \text{in } C^0([0,T];V')\,,
\end{aligned}
\tag{4.42}
$$

$$
\begin{aligned}
& z_\mu \rightharpoonup^* z \quad \text{in } L^\infty(0,T;L^2(\Gamma_c)) \cap H^1(0,T;H^1(\Gamma_c)')\,, \\
& z_\mu \to z \quad \text{in } C^0([0,T];H^1(\Gamma_c)')
\end{aligned}
\tag{4.43}
$$

hold as $\mu \searrow 0$. Arguing in the very same way as in the proof of Proposition 4.4, we find that the eight-uple $(w, \vartheta, z, \vartheta_s, \mathbf{u}, \chi, \xi, \boldsymbol{\eta})$ satisfies equations (2.32), (2.34), (2.36)–(2.40). Furthermore, combining (4.3), (4.8), (4.25), (4.42) and (4.43), we conclude that the quadruple $(w, z, \mathbf{u}, \chi)$ complies with the initial conditions (2.28)–(2.31).

Moreover, as already pointed out in [13, Sec. 4] (again on the basis of [2, Lemma 1.3, p. 42]), to conclude (2.33) it is sufficient to prove

$$
\limsup_{\mu \searrow 0} \int_0^T \int_\Omega w_\mu \vartheta_\mu \le \int_0^T \int_\Omega w\vartheta
\tag{4.44}
$$

(analogously for proving (2.35)). This can be shown by arguing in the very same way as for (4.34), so we refer the reader to the calculations developed in the proof of Proposition 4.4.

Finally, we recall that estimate (4.31) holds for a constant independent of $\mu > 0$. Further, we note that, by definition of $\mathscr{L}_\mu$ and convergences (4.13) and (4.42), there holds

$$
\gamma_\mu(w_\mu) = \vartheta_\mu - \mu w_\mu \rightharpoonup \vartheta \qquad \text{in } L^2(0,T;H)\,,
$$

so that, by the definition (3.2) of γ_μ, $w_\mu - \rho_\mu(w_\mu) \to 0$ in $L^2(0,T;H)$. Hence, in view of (4.42)

$$
\rho_\mu(w_\mu) \rightharpoonup w \qquad \text{in } L^2(0,T;H)\,.
\tag{4.45}
$$

Using that, by (3.5), $j_\mu^*(w_\mu) \ge j^*(\rho_\mu(w_\mu))$ a.e. in Ω, we conclude that for all $t_0 \in (0,T)$ and $r > 0$ such that $(t_0 - r, t_0 + r) \subset (0,T)$ there holds

$$
\int_{t_0-r}^{t_0+r} j^*(w) \le \liminf_{\mu \searrow 0} \int_{t_0-r}^{t_0+r} j^*(\rho_\mu(w_\mu)) \le \liminf_{\mu \searrow 0} \int_{t_0-r}^{t_0+r} j_\mu^*(w_\mu) \le 2r\overline{C}\,,
$$

where again the first inequality follows from (4.45) and weak lower semicontinuity of the integral functional induced by j^*. With the same Lebesgue point argument used for proving (4.31), we infer (2.20) (a completely analogous argument yields (2.23)). In the end, we point out that (2.20) ((2.23), respectively), (2.33) ((2.35), resp.), and (2.H2) yield an estimate for ϑ in $L^\infty(0,T;L^1(\Omega))$ (for ϑ_s in $L^\infty(0,T;L^1(\Gamma_c))$, resp.). □

A Appendix

The following result shows how the coercivity property (2.H2) translates in terms of the Yosida approximation of the functional j^*.

Lemma A.1. *Assume* (2.H1)–(2.H2). *Then,*

1. *there exists a constant* $\overline{C}_2 > 0$ *(*C_1 *being the same constant as in* (2.H2)*) such that*

$$\forall\, \mu > 0,\ u \in V\ :\quad \mu\|\mathscr{L}_\mu(u)\|_H^2 + j^*_\mu\left(\mathscr{L}_\mu(u)\right) \geq C_1\|u\|_{L^1(\Omega)} - \overline{C}_2\,; \tag{A.1}$$

2. *there exists a constant* $\overline{C}_2^* > 0$ *(*C_1 *being the same constant as in* (2.H2)*) such that*

$$\forall\, \mu > 0,\ v \in H^1(\Gamma_c)\ :\quad \mu\|\mathscr{L}_\mu(v)\|_{L^2(\Gamma_c)}^2 + j^*_\mu\left(\mathscr{L}_\mu(v)\right) \geq C_1\|v\|_{L^1(\Gamma_c)} - \overline{C}_2^*\,. \tag{A.2}$$

Proof. We shall just prove (A.1), the proof of (A.2) being completely analogous. For a given $u \in V$, let us put $w := \mathscr{L}_\mu(u)$: it follows from the definition of $\mathscr{L}_\mu$ and from (3.2) that $u - \mu w \in \gamma(\rho_\mu(w))$, whence

$$\rho_\mu(w) \in \ell\,(u - \mu w)\,. \tag{A.3}$$

Therefore, one has

$$\begin{aligned} j^*_\mu(w) \geq j^*\left(\rho_\mu(w)\right) &\geq C_1\|u - \mu w\|_{L^1(\Omega)} - C_2 \\ &\geq C_1\|u\|_{L^1(\Omega)} - \mu|\Omega|^{1/2}\|w\|_H - C_2 \\ &\geq C_1\|u\|_{L^1(\Omega)} - \mu\|w\|_H^2 - \frac{\mu}{4}C^2 - C_2\,, \end{aligned} \tag{A.4}$$

where the first inequality follows from (3.5), the second one from (2.H2), the third one from the Hölder inequality, and the last one from trivial computations. Hence, we conclude (A.1). □

We may now give the
Proof of Lemma 4.1. We shall just develop the construction of the sequences $\{w^0_\mu\}$ and $\{\vartheta^0_\mu\}$, the proof of the second part of the statement being completely analogous. For every $\mu > 0$, we define $w^0_\mu \in V$ as the solution of the variational equation

$$\int_\Omega w^0_\mu v + \mu \int_\Omega \nabla w^0_\mu \nabla v = \int_\Omega w_0 v \qquad \text{for all } v \in V.$$

Arguing in the very same way as in the proof of [3, Lemma 2.4], we find that (4.1)–(4.3) hold. By (4.4), we have

$$\vartheta^0_\mu = \mu w^0_\mu + \gamma_\mu(w^0_\mu),$$

which ensures that ϑ^0_μ is in V as well, since γ_μ is Lipschitz continuous. Then, (4.5) immediately ensues. □

References

[1] K.T. Andrews, M. Shillor, S. Wright, and A. Klarbring. A dynamic thermoviscoelastic contact problem with friction and wear. *Internat. J. Engrg. Sci.*, 14, 1291–1309, 1997.

[2] V. Barbu. *Nonlinear Semigroups and Differential Equations in Banach Spaces.* Noordhoff, Leyden, 1976.

[3] V. Barbu, P. Colli, G. Gilardi, and M. Grasselli. Existence, uniqueness, and long-time behaviour for a nonlinear Volterra integrodifferential equation. *Differential Integral Equations*, 13, 1233–1262, 2000.

[4] E. Bonetti, G. Bonfanti, and R. Rossi. Global existence for a contact problem with adhesion. *Math. Meth. Appl. Sci.*, 31, 1029–1064, 2008.

[5] E. Bonetti, G. Bonfanti, and R. Rossi. Well-posedness and long-time behaviour for a model of contact with adhesion. *Indiana Univ. Math. J.*, 56, 2787–2820, 2007.

[6] E. Bonetti, G. Bonfanti, and R. Rossi. Long-time behaviour of a thermomechanical model for adhesive contact. Preprint: Quaderno del Seminario Matematico di Brescia 14/2009, 1–37, 2009. To appear in *Discrete Contin. Dyn. Syst. Ser. S.*

[7] E. Bonetti, P. Colli, M. Fabrizio, and G. Gilardi. Global solution to a singular integrodifferential system related to the entropy balance. *Nonlinear Anal.*, 66, 1949–1979, 2007.

[8] E. Bonetti, P. Colli, M. Fabrizio, and G. Gilardi. Modelling and long-time behaviour for phase transitions with entropy balance and thermal memory conductivity. *Discrete Contin. Dyn. Syst. Ser. B*, 6, 1001–1026, 2006.

[9] E. Bonetti, P. Colli, M. Fabrizio, and G. Gilardi. Existence and boundedness of solutions for a singular phase field system. Preprint arXiv:0804.1628v1 [math.AP], 1–33, 2008.

[10] E. Bonetti, P. Colli, and M. Frémond. A phase field model with thermal memory governed by the entropy balance. *Math. Models Methods Appl. Sci.*, 13, 1565–1588, 2003.

[11] E. Bonetti, M. Frémond, and E. Rocca. A new dual approach for a class of phase transitions with memory: existence and long-time behaviour of solutions. *J. Math. Pures Appl.*, 88, 455–481, 2007.

[12] E. Bonetti and G. Schimperna. Local existence for Frémond's model of damage in elastic materials. *Contin. Mech. Thermodyn.*, 16, 319–335, 2004.

[13] G. Bonfanti, M. Frémond, and F. Luterotti. Global solution to a nonlinear system for irreversible phase changes. *Adv. Math. Sci. Appl.*, 10, 1–24, 2000.

[14] H. Brézis. *Opérateurs Maximaux Monotones et Semi-groupes de Contractions dans les Espaces de Hilbert.* Number 5 in North Holland Math. Studies. North-Holland, Amsterdam, 1973.

[15] M. A. Bruneaux and K. Benzarti. Durability of adhesively bonded assemblies: Mechanical and physico-chemical modeling. ERLPC n OA 55, Paris, LCPC Editions, 2008.

[16] G. Caginalp. An analysis of a phase field model of a free boundary. *Arch. Rational Mech. Anal.*, 92, 205–245, 1986.

[17] E. Di Benedetto and R. E. Showalter. Implicit degenerate evolution equations and applications. *SIAM J. Math. Anal.*, 12, 731–751, 1981.

[18] G. Duvaut and J.L. Lions. *Inequalities in Mechanics and Physics.* Springer-Verlag, Heidelberg-New York, 1976.

[19] C. Eck, J. Jiří, and M. Krbec. *Unilateral contact problems.* Pure and Applied Mathematics (Boca Raton) Vol. 270, Chapman & Hall/CRC, Boca Raton, Florida, 2005.

[20] I. Figueiredo and L. Trabucho. A class of contact and friction dynamic problems in thermoelasticity and in thermoviscoelasticity. *Internat. J. Engrg. Sci.*, 1, 45–66, 1995.

[21] M. Frémond. *Non-smooth Thermomechanics.* Springer-Verlag, Berlin, 2002.

[22] M. Frémond and B. Nedjar. Damage, gradient of damage and priciple of virtual power. *Internat. J. Solids Structures*, 33, 1083–1103, 1996.

[23] L. Nirenberg. On elliptic partial differential equations. *Ann. Scuola Norm. Sup. Pisa (3)*, **13**, 115–162, 1959.

[24] N. Point. Unilateral contact with adherence. *Math. Methods Appl. Sci.*, 10, 367–381, 1988.

[25] M. Raous, L. Cangémi, and M. Cocu. A consistent model coupling adhesion, friction, and unilateral contact. *Comput. Methods Appl. Mech. Eng.*, 177, 383–399, 1999.

[26] P. Shi and M. Shillor. Existence of a solution to the N-dimensional problem of thermoelastic contact. *Comm. Partial Differential Equations*, 9-10, 1597–1618, 1992.

[27] M. Shillor, M. Sofonea, and J. J. Telega. Models and analysis of quasistatic contact. Lecture Notes in Physics, 655. Springer, Berlin, 2004.

[28] J. Simon. Compact sets in the space $L^p(0,T;B)$. *Ann. Mat. Pura Appl. (4)*, 146, 65–96, 1987.

[29] M. Sofonea, W. Han, and M. Shillor. Analysis and approximation of contact problems with adhesion or damage. Pure and Applied Mathematics (Boca Raton), 276. Chapman & Hall/CRC, Boca Raton, FL, 2006.

[30] J.N. Theillout. Renforcement et réparation des ouvrages d'art par la technique des tôles collées. Thèse de l'Ecole Nationale des Ponts et Chaussées, Paris, 1983.

[31] J.M. Truong Dinh Tien. Contact avec adhèrence. Thèse de l'Ecole Nationale des Ponts et Chaussées, Paris, 1990.