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EXISTENCE AND ASYMPTOTIC ANALYSIS OF A PHASE FIELD MODEL FOR SUPERCOOLING

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Abstract. We prove an existence result for an initial-boundary value problem which models a perturbation of a phase transition phenomenon with supercooling effects. When the perturbation parameter goes to 0, an asymptotic analysis is performed. It leads to an existence result, in the framework of Young measures, for a slight modification of the original problem.

1. Introduction.

We address the following system of phase field type

$$\partial_t\vartheta + L\partial_t\chi - \kappa\Delta\vartheta = f \quad \text{in } \Omega \times (0,T), \tag{1.1}$$

$$-\eta(\vartheta,\nabla\chi)(\partial_t\chi)^- - \Delta\chi + \beta(\chi) + \sigma'(\chi) \ni \frac{L}{\vartheta_{\mathrm{c}}}(\vartheta - \vartheta_{\mathrm{c}}) \quad \text{in } \Omega \times (0,T), \tag{1.2}$$

(the symbol $(r)^-$ denoting the negative part of a number $r \in \mathbb{R}$), where Ω is a bounded, connected domain of $\mathbb{R}^N$, $N = 1,2,3$, with smooth boundary $\Gamma := \partial\Omega$, occupied by a physical system which undergoes a solid-liquid phase transition in the time interval $(0,T)$. We denote by Q the space-time cylinder $\Omega \times (0,T)$. The evolution of the phase change phenomenon is described in terms of the absolute temperature ϑ of the system (ϑ_{c} being the melting temperature), and of the order parameter χ, representing the volume fraction of the liquid phase. Hence, (1.1) is an energy balance equation, obtained by adopting the Fourier law $\mathbf{q} := -\kappa\nabla\vartheta$, with $\kappa > 0$, for the heat flux; $L > 0$ is the density of the

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latent heat of the phase transition, and f possibly stands for a heat source. On the other hand, the parabolic equation (1.2) yields the dynamics of the phase parameter: here, $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ is a maximal monotone operator, the subdifferential of a convex function $\widehat{\beta}$, while σ' is a Lipschitz continuous function. For example, we might choose $\beta := \partial I_{[a,b]}$, i.e., the subdifferential of the indicator function of the interval $[a, b]$, thus inducing a constraint on the values of χ. Combining this with an appropriate quadratic polynomial as function σ, $\widehat{\beta} + \sigma$ is equal to the *double obstacle potential*

$$\mathcal{O}(s) := \begin{cases} -(s-a)(s-b), & \text{if} \quad s \in [a,b], \\ +\infty, & \text{otherwise.} \end{cases} \tag{1.3}$$

On the other hand, β is also often chosen to be an increasing polynomial function, so that the sum $\beta + \sigma'$ yields the derivative of a nonconvex energy potential $\mathcal{W}$; e.g., the double well potential

$$\mathcal{W}(r) := (r^2 - 1)^2/4 \quad \forall r \in \mathbb{R}. \tag{1.4}$$

Finally, $\eta : \mathbb{R} \times \mathbb{R}^3 \to [0, +\infty)$ is a relaxation parameter function, which was first introduced in the modelling of solid-liquid phase transitions with supercooling effects in [10].

In fact, in the previous paper [10], the following phase field model was addressed:

$$\partial_t \vartheta + L \partial_t \chi - \kappa \Delta \vartheta = f \quad \text{in } \Omega \times (0,T), \tag{1.5}$$

$$\eta(\vartheta, \nabla\chi)\partial_t \chi - \Delta \chi + \partial I_{[0,1]}(\chi) \ni \frac{L}{\vartheta_{\mathrm{c}}}(\vartheta - \vartheta_{\mathrm{c}}) \quad \text{in } \Omega \times (0,T), \tag{1.6}$$

which was shown to be related to a generalized Stefan model with supercooling effects. A thermomechanical derivation, according to the approach proposed by M. Frémond (see [13]), was also developed for (1.5, 1.6). In addition, in [10] (1.5, 1.6) was also derived as an approximation of the Stefan model. Let us point out that such a derivation gives insight on the role of the relaxation parameter function η in (1.6); actually, η provides a continuous approximation of the map $(\vartheta, \nabla\chi) \mapsto c(\vartheta)/|\nabla\chi|$, where the function $c : \mathbb{R} \to [0, +\infty)$ describes the dependence of the normal velocity of the freezing line on the temperature. Hence, following the discussion in [10], we may think of

$$\eta(\vartheta, \nabla\chi) := \frac{c(\vartheta)}{|\nabla\chi| + \delta}, \quad \text{or} \quad \eta(\vartheta, \nabla\chi) := \frac{c(\vartheta)}{\sqrt{|\nabla\chi|^2 + \delta}},$$

for some $\delta > 0$. In [10], two existence results under two different sets of assumptions on η were proved for the system (1.5, 1.6), supplemented with third type boundary conditions on ϑ, homogeneous Neumann boundary conditions on χ, and suitable initial conditions on ϑ and χ.

Later on, in [15] it was argued that the order parameter equation (1.6) might be replaced by the following relaxed equation:

$$\varepsilon \partial_t \chi - \eta(\vartheta, \nabla\chi)(\partial_t \chi)^- - \Delta\chi + \partial I_{[0,1]}(\chi) \ni \frac{L}{\vartheta_{\mathrm{c}}}(\vartheta - \vartheta_{\mathrm{c}}) \quad \text{in } \Omega \times (0,T), \tag{1.7}$$

where $\varepsilon > 0$ is a fixed constant. In [15], it is indeed shown that the system (1.1, 1.7) provides an approximation of a generalized Stefan problem modelling a solid-liquid transition in which the water can stay liquid for some time before freezing also at temperatures below the melting temperature ϑ_c, but the ice melts at ϑ_c, in agreement with the physical experience.

Actually, in the present paper we will consider the PDE system coupling (1.1) and an alternative equation for the phase parameter, namely

$$\varepsilon\partial_t\chi - \eta(\vartheta, \nabla\chi)(\partial_t\chi)^- - \Delta\chi + \beta(\chi) + \sigma'(\chi) \ni \frac{L}{\vartheta_c}(\vartheta - \vartheta_c) \quad \text{in } \Omega \times (0, T), \tag{1.8}$$

(which of course generalizes (1.7)). Then, note that (1.2) can be formally obtained from (1.8) by setting $\varepsilon = 0$. More precisely, we will first prove an existence result for the system (1.1, 1.8), supplemented with the initial conditions

$$\vartheta(\cdot, 0) = \vartheta_0 \quad \chi(\cdot, 0) = \chi_0 \quad \text{in } \Omega \tag{1.9}$$

on ϑ and χ, with third type boundary conditions on ϑ, and with homogeneous Neumann boundary conditions on χ,

$$\kappa\partial_n\vartheta + \omega\vartheta = g, \quad \partial_n\chi = 0 \quad \text{in } \Gamma \times (0, T), \tag{1.10}$$

where ω is a positive constant and $g : \Gamma \times (0, T) \to \mathbb{R}$ a given function, related to the external temperature. Secondly, we will perform an asymptotic analysis of (1.1, 1.8, 1.9, 1.10) for vanishing ε, and analyse the relations between the limiting system and system (1.1, 1.2) in view of Young measure theory.

Let us point out that the equation (1.8) for the phase parameter displays a *doubly nonlinear* structure. More specifically, the analysis of (1.8) is connected with the study of this abstract doubly nonlinear equation

$$u'(t) + \mathcal{B}(t)(u'(t)) + \partial\phi(u(t)) \ni \mathcal{F}(u(t)) \quad \text{in } H, \quad \text{for a.e. } t \in (0, T), \tag{1.11}$$

where H is a Hilbert space, $\{\mathcal{B}(t)\}_{t\in(0,T)}$ is a family of maximal monotone operators on H, $\partial\phi$ is the subdifferential (in the sense of convex analysis) of a proper, convex, and l.s.c. functional $\phi : H \to (-\infty, +\infty]$, and, finally, $\mathcal{F} : H \to H$ is a given operator. In fact, setting $H := L^2(\Omega)$, it is straightforward to check that (1.8) may be rephrased in the form (1.11) with appropriate choices of $\{\mathcal{B}(t)\}_{t\in(0,T)}$, ϕ, and $\mathcal{F}$.

Therefore, the analysis of the system (1.1, 1.8) has led us to establish an existence theorem for the Cauchy problem associated with (1.11) in the aforementioned setup, and under the assumption that $\mathcal{F} : H \to H$ is a continuous operator with linear growth (cf. hypothesis (3.6) later on). Indeed, we may think of $\mathcal{F}$ as a *Lipschitz perturbation.* As for $\{\mathcal{B}(t)\}_{t\in(0,T)}$, we focus on the case of operators given by the product of a positive function α in $L^\infty(Q)$ and a maximal monotone bounded operator in H (see (3.9) below).

Doubly nonlinear equations of this kind are particularly relevant in the applications, as shown in [11]; nonetheless, let us point out that, as far as we know, (1.11) has not been investigated yet. Indeed, results in the case of a *time-independent* $\mathcal{B}$ and $\mathcal{F} \equiv 0$ (but with a more general operator $\partial\psi$ acting on u'), have been obtained in the seminal papers [11, 9] by means of the theory of maximal monotone operators, see [7, 8]. More recently, a Lipschitz continuous perturbation of a very particular type (but with a *time-independent* $\mathcal{B}$), has been tackled in [18], while the papers [1, 2, 3] are concerned with the challenging analysis of a class of doubly nonlinear equations in which the subdifferential operator on the time derivative depends on the unknown itself.

The plan of the paper is as follows. In the following section we give the notation, the assumptions, and state the main results. In Section 3, we prove our existence theorem for (the Cauchy problem related to) (1.11) by exploiting an approximation technique, based on a time discretization procedure. Subsequently, in Section 4, we develop the proof of our existence result for problem (1.1, 1.8, 1.9, 1.10). More precisely, we approximate the system (1.1, 1.8) by introducing the Yosida regularization of the operator β, obtain an existence result for the latter approximate system by means of a fixed point procedure (which relies on the results of Section 3), and then we pass to the limit with respect to the regularization parameter. The asymptotic analysis of (1.1, 1.8, 1.9, 1.10), as $\varepsilon \to 0$, is performed in Section 5 in the framework of Young measures. Finally, for the reader's convenience we recall some useful results in the Appendix.

2. General setup and main results.

Our functional setting is given by the spaces

$$H := L^2(\Omega), \quad V := H^1(\Omega), \quad \text{and} \quad W := \left\{ v \in H^2(\Omega) \, : \, \partial_n v = 0 \right\};$$

we identify H with its dual space H', so that $W \subset V \subset H \subset V' \subset W'$, with dense and compact embeddings. We denote by $\| \cdot \|_V$, $\| \cdot \|_H$ and $\| \cdot \|_{V'}$ the norms on V, H, and V', respectively, and by $(\cdot, \cdot)_H$ the scalar product in H, while $\langle \cdot, \cdot \rangle$ is the duality pairing between V' and V. In general, given a Banach space Y, $C^0_w([0,T];Y)$ will denote the space of the weakly continuous Y-valued functions on $[0,T]$. Finally, we denote by $C_0(Q)$ the space of the continuous functions on Q with compact support.

Assumptions on the data. We assume that the relaxation parameter function η fulfils the following:

$$\eta : \mathbb{R} \times \mathbb{R}^3 \to [0, +\infty) \quad \text{is continuous;} \tag{2.1}$$

$$\exists K_\eta > 0 \quad \eta(u,v) \le K_\eta \quad \forall (u,v) \in \mathbb{R} \times \mathbb{R}^3; \tag{2.2}$$

$$\exists k_\eta > 0 \quad \eta(u,v) \ge \frac{k_\eta}{1+|v|} \quad \forall (u,v) \in \mathbb{R} \times \mathbb{R}^3. \tag{2.3}$$

Moreover,

$$\beta : \mathbb{R} \to 2^{\mathbb{R}} \text{ is a maximal monotone graph, } 0 \in \beta(0), \text{ and } \beta = \partial\widehat{\beta}, \text{ with} \tag{2.4}$$

$$\widehat{\beta} : \mathbb{R} \to [0, \infty] \text{ convex, l.s.c.;} \tag{2.5}$$

$$\sigma \in C^1(\mathbb{R}), \text{ and } \sigma' \in C^{\mathrm{Lip}}(\mathbb{R}) \text{ with Lipschitz constant } \Lambda_\sigma. \tag{2.6}$$

Owing to (2.4)–(2.5) $\widehat{\beta}(r) \ge \widehat{\beta}(0)$ for all $r \in \mathbb{R}$, so that up to a translation we have

$$\widehat{\beta}(r) \ge 0 \quad \forall \, r \in \mathbb{R}. \tag{2.7}$$

The graph $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ and the function $\widehat{\beta} : \mathbb{R} \to [0, \infty]$ induce a maximal monotone operator $\beta_H : H \to 2^H$ and a functional $\widehat{\beta}_H : H \to [0, \infty]$, with $\beta_H = \partial \widehat{\beta}_H$. In the sequel, we will often employ the notation $D(\widehat{\beta}_H)$ for the proper domain of $\widehat{\beta}_H$.

Finally, when needed we will also strengthen our coercivity assumptions on the sum $\widehat{\beta} + \sigma$ by requiring that

$$\exists C_{\beta,1}, \; C_{\beta,2} \ge 0 \quad \text{such that} \quad \widehat{\beta}(s) + \sigma(s) \ge C_{\beta,1}|s|^2 - C_{\beta,2} \quad \forall \; s \in D(\widehat{\beta}). \tag{2.8}$$

Remark 2.1. Note that, if $\widehat{\beta}$ and σ are polynomial functions, and the degree of $\widehat{\beta}$ is bigger than the degree of σ, then (2.8) clearly holds. So, the choice $\widehat{\beta}+\sigma=\mathcal{W}$, with $\mathcal{W}$ the standard double-well potential (1.4), is admissible.

Another admissible choice (associated with the original problem (1.5)–(1.6)), is given by $\widehat{\beta}$ being any proper, convex, l.s.c. functional with bounded domain (like the indicator function of $[0,1]$), and σ being any function satisfying (2.6): for example, the double-obstacle potential (1.3) complies with these requirements. Moreover, in this framework also the logarithmic functional $\mathcal{L}$, defined by $\mathcal{L}(s) := s\ln(s)+(1-s)\ln(1-s)$ if $s\in(0,1)$ and $\mathcal{L}(s) := +\infty$ otherwise, would be an admissible choice for $\widehat{\beta}=\mathcal{L}$.

As far as the data of the problem are concerned, we suppose that

$$\vartheta_0 \in H, \quad \chi_0 \in V \cap D(\widehat{\beta}_H); \tag{2.9}$$

$$f \in L^2(0,T;V'), \quad g \in L^2(0,T;H^{-1/2}(\Gamma)). \tag{2.10}$$

2.1. *Variational formulation of the problem and existence result.* Let us introduce the operator $A : V \to V'$ by

$$\langle Au, v\rangle := \int_\Omega \nabla u \cdot \nabla v \, dx \quad \forall u,v \in V,$$

and let us also consider $J : V \to V'$, defined by

$$\langle Ju, v\rangle := \int_\Omega \nabla u \cdot \nabla v + \omega \langle u, v\rangle_\Gamma \quad \forall u,v \in V. \tag{2.11}$$

Of course, J is linear and bounded on V; moreover, a standard version of Poincaré's inequality ensures that the operator J is also coercive on V, with bounded inverse $J^{-1} : V' \to V$. Thus, we will endow the spaces V and V' with the norms

$$\|v\|_V^2 := \langle Jv, v\rangle \quad \forall v \in V, \quad \|w\|_{V'}^2 := \langle w, J^{-1}(w)\rangle \quad \forall w \in V', \tag{2.12}$$

which are equivalent to the usual norms on V and V'.

We also consider the function $F \in L^2(0,T;V')$ given by

$$\langle F(t), v\rangle := \langle f(t), v\rangle + \langle g(t), v\rangle_\Gamma, \quad \forall v \in V \text{ for a.e. } t \in (0,T). \tag{2.13}$$

In the present framework, we can give the variational formulation for the initial boundary value problem (1.1, 1.8, 1.9, 1.10). Note that for convenience we set most of the constants equal to 1 and we incorporate the constant $(L/\vartheta_c)\vartheta_c = L$ in (1.8) in the term $\sigma'(\chi)$, while highlighting the coefficient ε of $\partial_t\chi$ in (1.8) in view of a subsequent asymptotic analysis.

Problem 2.2. Find $\vartheta \in H^1(0,T;V') \cap C^0([0,T];H) \cap L^2(0,T;V)$, $\chi \in H^1(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;W)$, such that $\chi \in D(\widehat{\beta})$ a.e. in Q, and

$$\partial_t\vartheta + \partial_t\chi + J\vartheta = F \quad \text{in } V' \quad \text{a.e. in } (0,T), \tag{2.14}$$

$$\begin{aligned} &\varepsilon\partial_t\chi - \eta(\vartheta,\nabla\chi)(\partial_t\chi)^- + A\chi + \xi + \sigma'(\chi) = \vartheta \quad \text{in } H, \quad \text{a.e. in } (0,T), \\ &\qquad\qquad \text{for some } \xi \in L^2(0,T;H) \text{ with } \xi \in \beta(\chi) \text{ a.e. in } Q, \end{aligned} \tag{2.15}$$

$$\vartheta(x,0) = \vartheta_0(x), \quad \chi(x,0) = \chi_0(x) \quad \text{for a.e. } x \in \Omega. \tag{2.16}$$

We can now state our main existence result.

Theorem 2.3. Assume (2.1)–(2.2), (2.4)–(2.6), and (2.9)–(2.10). Then, Problem 2.2 admits a solution (ϑ,χ,ξ).

Remark 2.4. Let us stress that the coercivity assumptions (2.3) on η and (2.8) are not needed in the proof of Theorem 2.3, but instead play a crucial role in the proof of Theorem 2.7 stated below. As it will be clear from the proof of the latter results, (2.3) and (2.8) basically compensate for the poorness of estimates on $\partial_t \chi$.

Remark 2.5. Because of the special doubly nonlinear character of (2.15) (in particular, due to the problems arising from the the factor $\eta(\vartheta, \nabla\chi)$ and the nonlinearity $\beta(\chi)$), we could not obtain any uniqueness result for the Problem 2.2.

2.2. *Singular limit of Problem* 2.2. Let $(\vartheta_0, \chi_0, f, g)$ be a quadruple of data complying with (2.9) and (2.10), and let $\{\vartheta_0^\varepsilon\}_\varepsilon$, $\{\chi_0^\varepsilon\}_\varepsilon$, $\{f^\varepsilon\}_\varepsilon$, and $\{g^\varepsilon\}_\varepsilon$ be suitable approximating sequences as $\varepsilon \downarrow 0$, fulfilling

$$\chi_0^\varepsilon \rightharpoonup \chi_0 \quad \text{in } V, \quad \sup_\varepsilon \left|\widehat{\beta}_H(\chi_0^\varepsilon)\right| < \infty, \quad \vartheta_0^\varepsilon \rightharpoonup \vartheta_0 \quad \text{in } H, \tag{2.17}$$

$$f^\varepsilon \rightharpoonup f \quad \text{in } L^2(0,T;V'), \quad g^\varepsilon \rightharpoonup g \quad \text{in } L^2(0,T;H^{-1/2}(\Gamma)), \tag{2.18}$$

so that the sequence $\{F^\varepsilon\} \subset L^2(0,T;V')$ defined by $\{f^\varepsilon\}$ and $\{g^\varepsilon\}$ by means of (2.13) also fulfills

$$F^\varepsilon \rightharpoonup F \quad \text{in } L^2(0,T;V') \text{ as } \varepsilon \downarrow 0. \tag{2.19}$$

Remark 2.6. The boundedness assumption (2.17) for $\widehat{\beta}_H(\chi_0^\varepsilon)$ follows from the convergence for χ_0^ε if a condition of the form

$$\exists C_{\beta,3} \geq 0,\ q > 0, \quad \text{such that} \quad \widehat{\beta}_H(v) \leq C_{\beta,3}\left(\|v\|_V^q + 1\right) \quad \forall v \in D(\widehat{\beta}_H) \cap V \tag{2.20}$$

holds. The above estimate is, for example, satisfied if $\widehat{\beta}$ is polynomial of at most degree 6 or if $\widehat{\beta}$ is an indicator function.

Theorem 2.7. Assume (2.1)–(2.8). Let $\{\vartheta_0^\varepsilon\}_\varepsilon$, $\{\chi_0^\varepsilon\}_\varepsilon$, $\{f^\varepsilon\}_\varepsilon$, and $\{g^\varepsilon\}_\varepsilon$ fulfill (2.17)–(2.18) and, accordingly, let $\{(\vartheta_\varepsilon, \chi_\varepsilon, \xi_\varepsilon)\}$ be a sequence of solutions to Problem 2.2 supplemented with the sequence of data $\{(\vartheta_0^\varepsilon, \chi_0^\varepsilon, f^\varepsilon, g^\varepsilon)\}$; for all $\varepsilon > 0$ set $e_\varepsilon := \vartheta_\varepsilon + \chi_\varepsilon$.

Then, there exist subsequences $\{\vartheta_{\varepsilon_k}\}$, $\{\chi_{\varepsilon_k}\}$, $\{\xi_{\varepsilon_k}\}$, and $\vartheta \in L^\infty(0,T;H) \cap L^2(0,T;V)$, $\chi \in L^\infty(0,T;V) \cap L^2(0,T;W)$, with $e := \vartheta + \chi \in H^1(0,T;V') \cap C^0_w([0,T];H)$, $\xi \in L^2(0,T;H)$, and a Young measure (see Appendix B) $\boldsymbol{\nu} = \{\nu_{(x,t)}\} \in \mathcal{Y}(Q;\mathbb{R})$, with

$$\operatorname{supp}(\nu_{(x,t)}) \subset \cap_{p=1}^\infty \overline{\{\partial_t \chi_{\varepsilon_k}(x,t) : k \geq p\}} \quad \text{for a.e. } (x,t) \in Q, \tag{2.21}$$

and, setting

$$\ell(x,t) := \int_{\mathbb{R}} (\lambda)^- d\nu_{(x,t)}(\lambda) \quad \text{for a.e. } (x,t) \in Q, \tag{2.22}$$

we have $\ell \in L^2(0,T;L^{4/3}(\Omega))$, such that the following convergences hold as $k \uparrow \infty$:

$$\chi_{\varepsilon_k} \rightharpoonup^* \chi \quad \text{in } L^\infty(0,T;V) \cap L^2(0,T;W), \tag{2.23}$$

$$\chi_{\varepsilon_k} \to \chi \quad \text{in } L^p(0,T;V) \text{ for all } 1 \le p < \infty, \tag{2.24}$$

$$\varepsilon \partial_t \chi_\varepsilon \to 0 \quad \text{in } L^2(0,T;L^2(\Omega)) \text{ as } \varepsilon \downarrow 0, \tag{2.25}$$

$$(\partial_t \chi_{\varepsilon_k})^- \rightharpoonup \ell \quad \text{in } L^2(0,T;L^{4/3}(\Omega)), \tag{2.26}$$

$$\vartheta_{\varepsilon_k} \rightharpoonup^* \vartheta \quad \text{in } L^\infty(0,T;H) \cap L^2(0,T;V), \tag{2.27}$$

$$\vartheta_{\varepsilon_k} \to \vartheta \quad \text{in } L^p(0,T;H) \text{ for all } 1 \le p < \infty, \tag{2.28}$$

$$e_{\varepsilon_k} \rightharpoonup \vartheta + \chi \quad \text{in } H^1(0,T;V'), \tag{2.29}$$

$$e_{\varepsilon_k} \to \vartheta + \chi \quad \text{in } C^0([0,T];V') \cap L^p(0,T;H) \text{ for all } 1 \le p < \infty, \tag{2.30}$$

$$\xi_{\varepsilon_k} \rightharpoonup \xi \quad \text{in } L^2(0,T;H). \tag{2.31}$$

Moreover, the quadruple $(\vartheta, \chi, \xi, \ell)$ fulfills (2.14), the initial condition

$$e(x,0) = \vartheta_0(x) + \chi_0(x) \quad \text{for a.e. } x \in \Omega, \tag{2.32}$$

and

$$\begin{aligned} -\eta(\vartheta, \nabla\chi)\ell + A\chi + \xi + \sigma'(\chi) = \vartheta \quad \text{in } H, \quad \text{a.e. in } (0,T), \\ \xi \in \beta(\chi) \quad \text{a.e. in } Q. \end{aligned} \tag{2.33}$$

Finally, for all $0 \le t_1 < t_2 \le T$ there holds

$$\chi(x,t_1) - \int_{t_1}^{t_2} \ell(x,t)dt \le \chi(x,t_2) \quad \text{for a.e.} \quad x \in \Omega. \tag{2.34}$$

More generally, let $\mu \in M(Q)$ the limit Radon measure of $\partial_t \chi_{\varepsilon k}$ and ρ the Radon measure on Q given by

$$\langle \rho, f \rangle := \int_Q f(x,t) \left(\int_{\mathbb{R}} \xi d\nu_{(x,t)}(\xi) \right) dxdt \quad \forall f \in C_0(Q). \tag{2.35}$$

Then,

$$\langle \mu, f \rangle \ge \langle \rho, f \rangle \quad \forall f \in C_0(Q) \text{ with } f \ge 0. \tag{2.36}$$

In the sequel of the paper, we adopt the convention of denoting by the two symbols C, C' (whose meaning can vary within the same line) all the positive constants occurring in the estimates, in some cases specifying their dependence on other known constants.

REMARK 2.8. The inequality (2.34) yields that $-\ell$ is a lower bound for the decrease of χ. It is an open question whether one can formulate conditions ensuring that (2.34) becomes an equality on some subset of Ω and for some values of t and s. Indeed, so far we have not been able to conclude that $\ell = (\partial_t \chi)^-$, and hence to solve our original problem (1.1, 1.2, 1.9, 1.10).

3. An existence result for an abstract doubly nonlinear evolution equation.

As we have mentioned in the Introduction, the proof of Theorem 2.3 shall be carried out by means of a Schauder fixed point argument, which in fact relies on separate well-posedness results for the single equations (2.14) and (2.15). Due to the doubly nonlinear character of the latter equation, in that case existence shall follow from the main result

(Theorem 3.2 below) of this section, which is devoted to the analysis of the abstract doubly nonlinear evolution inclusion (1.11).

Let us now enlist our assumptions on the function α, on the operators B and $\mathcal{F}$, as well as on the functional ϕ. Namely, we suppose that (compare with the growth and coercivity assumptions of [11, 9]):

$$\exists K_\alpha > 0 \quad \text{s.t.} \quad 0 \leq \alpha(x,t) \leq K_\alpha \quad \text{for a.e. } (x,t) \in Q; \tag{3.1}$$

$$B : \mathbb{R} \to 2^{\mathbb{R}} \quad \text{is maximal monotone, } 0 \in B(0), \text{ and} \tag{3.2}$$

$$\exists \Psi > 0 \ : \quad |\xi| \leq \Psi(|v|+1) \quad \forall \xi \in B(v) \quad \forall v \in \mathbb{R}; \tag{3.3}$$

$$\begin{gathered} \phi : H \to (-\infty, +\infty] \quad \text{is proper, convex, l.s.c., and } \exists S \geq 0 \text{ s.t.} \\ \text{the functional } u \mapsto \phi(u) + S\|u\|_H^2 \text{ has compact sublevels;} \end{gathered} \tag{3.4}$$

$$\mathcal{F} : H \to H \quad \text{is a continuous operator, and} \tag{3.5}$$

$$\exists M > 0 \quad \|\mathcal{F}(u)\|_H \leq M\left(\|u\|_H + 1\right) \quad \forall u \in H. \tag{3.6}$$

For example, a *Lipschitz continuous* operator $\mathcal{F}$ is admissible within this framework. Note also that, by convexity, there exist positive constants S' and C_ϕ such that

$$\phi(u) + S'\|u\|_H^2 \geq -C_\phi \quad \forall\ u \in H. \tag{3.7}$$

We will denote by B_H the realization of the operator B on H. Hence, $B_H : H \to 2^H$ is a maximal monotone operator, fulfilling

$$\exists \Psi > 0 \ : \quad \|\xi\|_H \leq \Psi(\|v\|_H + |\Omega|^{1/2}) \quad \forall \xi \in B_H(v) \quad \forall v \in H. \tag{3.8}$$

Moreover, for a.e. $t \in (0,T)$ we will call $\mathcal{B}(t)$ the operator $\mathcal{B}(t) : H \to 2^H$ defined by

$$\begin{gathered} v \in \mathcal{B}(t)(u) \quad \text{if there exists } \xi \in H,\ \xi \in B_H(u) \\ \text{such that } v(x) = \alpha(x,t)\xi(x), \quad \text{for a.e. } x \in \Omega. \end{gathered} \tag{3.9}$$

Problem formulation. In view of the notation (3.9), we can now give a precise formulation to the Cauchy problem for (1.11).

PROBLEM 3.1. Given $u_0 \in H$ and $f \in L^2(0,T;H)$, find a function $u \in H^1(0,T;H)$ such that

$$u(0) = u_0, \tag{3.10}$$

and there exist $w,\ v \in L^2(0,T;H)$ such that

$$w(t) \in \mathcal{B}(t)(u'(t)) \quad \text{for a.e.}\, t \in (0,T), \tag{3.11}$$

$$v(t) \in \partial\phi(u(t)) \quad \text{for a.e. } t \in (0,T), \tag{3.12}$$

$$u'(t) + w(t) + v(t) = \mathcal{F}(u(t)) + f(t) \quad \text{for a.e. } t \in (0,T). \tag{3.13}$$

THEOREM 3.2. Assume (3.1)–(3.6). Then, for any $u_0 \in D(\phi)$ Problem 3.1 has a solution $u \in H^1(0,T;H)$.

As it will be clear from the proof of Theorem 3.2, we can suppose $f \equiv 0$ in (3.13) without loss of generality, since this does not alter the substance of the argument.

3.1. *Approximation.*

Time discretization. We fix a time step $\tau > 0$, such that there exists some $N_\tau \in \mathbb{N}$ with $\tau N_\tau = T$, and consider the corresponding partition of the interval $(0, T)$,

$$\mathcal{P}_\tau := \{t_0 = 0 < t_1 < \cdots < t_n < \cdots < t_{N_\tau - 1} < t_{N_\tau} = T\},$$
$$t_n := n\tau\,, \quad \text{for } n = 1, \dots, N_\tau.$$

We also set

$$\alpha_\tau^n(x) := \frac{1}{\tau}\int_{t_{n-1}}^{t_n} \alpha(x,t)dt \quad \text{for a.e. } x \in \Omega, \quad n = 1, \dots, N_\tau. \tag{3.14}$$

By (3.1), $\alpha_\tau^n \in L^\infty(\Omega)$ for all $n = 1, \dots, N_\tau$, so that the operator $\mathcal{B}_\tau^n : H \to 2^H$ defined by

$$\begin{aligned} v \in \mathcal{B}_\tau^n(u) \quad &\text{if there exists } \xi \in H,\, \xi \in B_H(u), \\ \text{s.t.} \quad &v(x) = \alpha_\tau^n(x)\xi(x) \text{ for a.e. } x \in \Omega, \end{aligned} \tag{3.15}$$

is well defined, maximal monotone, and bounded on H. Following the approach of [11, 9], the starting point for the construction of approximate solutions to Problem 3.1 is the following *backward finite difference scheme:*

Problem 3.3. Given $U_\tau^0 := u_0$, find $U_\tau^1, \dots, U_\tau^{N_\tau} \in H$, $w_\tau^1, \dots, w_\tau^{N_\tau} \in H$, and $v_\tau^1, \dots, v_\tau^{N_\tau} \in H$, such that for every $n = 1, \dots, N_\tau$,

$$\frac{u_\tau^n - u_\tau^{n-1}}{\tau} + w_\tau^n + v_\tau^n = \mathcal{F}(u_\tau^{n-1}) \quad \text{in } H, \tag{3.16}$$

$$w_\tau^n \in \mathcal{B}_\tau^n\left(\frac{u_\tau^n - u_\tau^{n-1}}{\tau}\right), \tag{3.17}$$

$$v_\tau^n \in \partial\phi(u_\tau^n). \tag{3.18}$$

Indeed, Problem 3.3 has at least one solution $\{(u_\tau^n, w_\tau^n, v_\tau^n)\}_{n=1}^{N_\tau}$. This can be shown by slightly adapting the proof of [9, Lemma 3.1].

Approximate solutions. Let $\overline{U}_\tau$ and $\underline{U}_\tau$ be, respectively, the left-continuous and the right-continuous piecewise-constant interpolant of the values $\{u_\tau^n\}_{n=1}^{N_\tau}$ fulfilling $\overline{U}_\tau(t_n) = \underline{U}_\tau(t_n) = u_\tau^n$ for all $n = 1, \dots, N_\tau$, i.e.,

$$\overline{U}_\tau(t) = u_\tau^n \quad \forall t \in (t_{n-1}, t_n], \qquad \underline{U}_\tau(t) = u_\tau^{n-1} \quad \forall t \in [t_{n-1}, t_n), \tag{3.19}$$

$n = 1, \dots, N_\tau$. We also introduce the piecewise linear interpolant U_τ of $\{u_\tau^n\}_{n=1}^{N_\tau}$, defined by

$$U_\tau(t) := \frac{t - t_{n-1}}{\tau} u_\tau^n + \frac{t_n - t}{\tau} u_\tau^{n-1} \quad \forall t \in [t_{n-1}, t_n), \quad n = 1, \dots, N_\tau. \tag{3.20}$$

Also, let $\overline{W}_\tau$ and $\overline{V}_\tau$ be the left-continuous piecewise constant interpolants of the values $\{w_\tau^n\}_{n=1}^{N_\tau}$ and $\{v_\tau^n\}_{n=1}^{N_\tau}$. Furthermore, we consider the piecewise constant interpolant $\overline{\alpha}_\tau$ of $\{\alpha_\tau^n(x)\}_{n=1}^{N_\tau}$, i.e.,

$$\text{for } t_{n-1} < t \le t_n \qquad \overline{\alpha}_\tau(x,t) := \alpha_\tau^n(x) \quad \text{for a.e. } x \in \Omega\,. \tag{3.21}$$

Note that $\overline{\alpha}_\tau \in L^\infty(Q)$ and for any $1 \le p < \infty$,

$$\overline{\alpha}_\tau \to \alpha \quad \text{in } L^p(Q) \ \text{ as } \tau \downarrow 0. \tag{3.22}$$

Accordingly, we introduce the family of operators $\overline{\mathcal{B}}_\tau(t) : H \to 2^H$ by setting

$$\begin{aligned} v \in \overline{\mathcal{B}}_\tau(t)(u) \quad &\text{if there exists } \xi \in H,\ \xi \in B_H(u) \text{ s.t.} \\ &v(x) = \overline{\alpha}_\tau(x,t)\xi(x), \quad \text{for a.e. } x \in \Omega. \end{aligned} \tag{3.23}$$

Hence, (3.16)–(3.18) may be rewritten as

$$U_\tau'(t) + \overline{W}_\tau(t) + \overline{V}_\tau(t) = \mathcal{F}(\underline{U}_\tau(t)) \quad \text{for a.e. } t \in (0,T), \tag{3.24}$$

$$\overline{W}_\tau(t) \in \overline{\mathcal{B}}_\tau(t)(U_\tau'(t)) \quad \text{for a.e. } t \in (0,T), \tag{3.25}$$

$$\overline{V}_\tau(t) \in \partial\phi(\overline{U}_\tau(t)) \quad \text{for a.e. } t \in (0,T). \tag{3.26}$$

Finally, let $\bar{\mathsf{t}}_\tau,\ \underline{\mathsf{t}}_\tau : [0,T] \to [0,T]$ be defined by

$$\begin{aligned} \bar{\mathsf{t}}_\tau(0) = \underline{\mathsf{t}}_\tau(0) := 0, \quad \bar{\mathsf{t}}_\tau(t) := t_k \quad &\text{for} \quad t \in (t_{k-1}, t_k], \\ \text{and} \quad \underline{\mathsf{t}}_\tau(t) := t_{k-1} \quad &\text{for} \quad t \in [t_{k-1}, t_k). \end{aligned} \tag{3.27}$$

Of course, for every $t \in [0,T]$ $\bar{\mathsf{t}}_\tau(t) \downarrow t$ and $\underline{\mathsf{t}}_\tau(t) \uparrow t$ as $\tau \downarrow 0$.

We will prove that, up to a subsequence, the sequence $\{(U_\tau, \overline{W}_\tau, \overline{V}_\tau)\}_\tau$ converges to a triplet (u,w,v) solving Problem 3.1.

Preliminary results. The following result, whose proof is immediate, will play a crucial role in passing to the limit in (3.24)–(3.26).

Lemma 3.4. Let $\{\alpha_m\} \subset L^\infty(Q)$ be a sequence fulfilling

$$\exists C \geq 0 \quad 0 \leq \alpha_m(x,t) \leq C \quad \text{for a.e. } (x,t) \in Q, \tag{3.28}$$

$$\exists \alpha \in L^\infty(Q) \quad \text{s.t.} \quad \alpha_m(x,t) \to \alpha(x,t) \quad \text{for a.e. } (x,t) \in Q. \tag{3.29}$$

For every $m \in \mathbb{N}$, let $\{\mathcal{B}_m(t)\}$ be the family of maximal monotone operators associated with α_m through (3.9). Let us denote by $\mathscr{B}_m$ the realization of $\{\mathcal{B}_m(t)\}$ on $L^2(0,T;H)$, i.e., the maximal monotone operator $\mathscr{B}_m : L^2(0,T;H) \to 2^{L^2(0,T;H)}$ defined by

$$v \in \mathscr{B}_m(u) \quad \Leftrightarrow \quad v(t) \in \mathcal{B}_m(t)(u(t)) \quad \text{for a.e. } t \in (0,T), \quad u,v \in L^2(0,T;H).$$

Analogously, let $\mathscr{B} : L^2(0,T;H) \to 2^{L^2(0,T;H)}$ be the operator associated with $\{\mathcal{B}(t)\}$ (cf. (3.9)).

Then, we have that (see Appendix A)

$$\mathscr{B}_m \ G\text{-converges to } \mathscr{B} \quad \text{in } L^2(0,T;H) \text{ as } m \uparrow \infty. \tag{3.30}$$

We will also need the following *Discrete Gronwall lemma* (see for example [14], Prop. 2.2.1.)

Lemma 3.5. Let $\psi, \alpha_0, \alpha_1, \dots, \alpha_n, x_0, x_1, \dots, x_n$ be given nonnegative numbers such that

$$x_0 \leq \psi, \quad x_i \leq \psi + \sum_{j=0}^{i-1} \alpha_j x_j, \ \forall 1 \leq i \leq n.$$

Then, we have

$$x_i \leq \psi \exp\left(\sum_{j=0}^{i-1} \alpha_j\right), \ \forall 1 \leq i \leq n.$$

3.2. *Proof of Theorem* 3.2.

A priori estimates on the approximate solutions. First, we test (3.16) by $u_\tau^k - u_\tau^{k-1}$. In view of (3.17), there exists $\xi_\tau^k \in B_H\left(\frac{u_\tau^k - u_\tau^{k-1}}{\tau}\right)$ such that $w_\tau^k(x) = \alpha_\tau^k(x)\xi_\tau^k(x)$ for a.e. $x \in \Omega$, hence

$$(w_\tau^k, u_\tau^k - u_\tau^{k-1})_H = \tau \int_\Omega \alpha_\tau^k(x)\xi_\tau^k(x)\left(\frac{u_\tau^k(x) - u_\tau^{k-1}(x)}{\tau}\right)\, dx \geq 0 \tag{3.31}$$

due to the fact that $\alpha_\tau^k \geq 0$ a.e. in Ω and to the assumption (3.2) on the operator $B : \mathbb{R} \to 2^{\mathbb{R}}$. Moreover, owing to the convexity inequality

$$(v_\tau^k, u_\tau^k - u_\tau^{k-1})_H \geq \phi(u_\tau^k) - \phi(u_\tau^{k-1}) \tag{3.32}$$

and to the trivial estimate

$$(\mathcal{F}(u_\tau^{k-1}), u_\tau^k - u_\tau^{k-1})_H \leq \frac{\tau}{2}\|\mathcal{F}(u_\tau^{k-1})\|_H^2 + \frac{\tau}{2}\left\|\frac{u_\tau^k - u_\tau^{k-1}}{\tau}\right\|_H^2,$$

testing (3.16) by $u_\tau^k - u_\tau^{k-1}$ leads to (cf. (3.6))

$$\begin{aligned}\frac{\|u_\tau^k - u_\tau^{k-1}\|_H^2}{2\tau} + \phi(u_\tau^k) &\leq \phi(u_\tau^{k-1}) + \frac{\tau}{2}\|\mathcal{F}(u_\tau^{k-1})\|_H^2 \\ &\leq \phi(u_\tau^{k-1}) + M^2\tau\left(1 + \|u_\tau^{k-1}\|_H^2\right).\end{aligned} \tag{3.33}$$

Arguing in the same way as in the proof of [17, Prop. 4.6], we note that

$$\begin{aligned}&\frac{1}{2}\|u_\tau^n\|_H^2 - \frac{1}{2}\|u_0\|_H^2 = \sum_{k=1}^n \left(\frac{1}{2}\|u_\tau^k\|_H^2 - \frac{1}{2}\|u_\tau^{k-1}\|_H^2\right) \\ &\leq \sum_{k=1}^n \left(\|u_\tau^k\|_H^2 - \|u_\tau^k\|_H\|u_\tau^{k-1}\|_H\right) \\ &\leq \sum_{k=1}^n \|u_\tau^k\|_H\|u_\tau^k - u_\tau^{k-1}\|_H \leq \mu\sum_{k=1}^n \frac{\|u_\tau^k - u_\tau^{k-1}\|_H^2}{2\tau} + \frac{1}{2\mu}\sum_{k=1}^n \tau\|u_\tau^k\|_H^2 \\ &\leq \sum_{k=1}^n \mu\Big(\phi(u_\tau^{k-1}) - \phi(u_\tau^k) + M^2\tau\|u_\tau^{k-1}\|_H^2\Big) + \mu M^2 T + \frac{1}{2\mu}\sum_{k=1}^n \tau\|u_\tau^k\|_H^2 \\ &\leq \mu\Big(\phi(u_0) - \phi(u_\tau^n)\Big) + \mu M^2(T + \tau\|u_0\|_H^2) + (\mu M^2\tau + \frac{\tau}{2\mu})\sum_{k=1}^n \tau\|u_\tau^k\|_H^2 \\ &\leq \mu S'\|u_\tau^n\|_H^2 + \mu\left(\phi(u_0) + C_\phi + M^2T + M^2\tau\|u_0\|_H^2\right) + (\mu M^2\tau + \frac{\tau}{2\mu})\sum_{k=1}^n \tau\|u_\tau^k\|_H^2,\end{aligned}$$

where we have used Young's inequality for a suitable $\mu > 0$ to be chosen in the fourth inequality, (3.33) in the fifth inequality, and finally (3.7). Hence, we obtain

$$\|u_\tau^n\|_H^2 \leq C + 2\mu S'\|u_\tau^n\|_H^2 + 2\left(\mu M^2\tau + \frac{\tau}{2\mu}\right)\sum_{k=1}^n \tau\|u_\tau^k\|_H^2,$$

where the constant C only depends on u_0, and the data of our problem. Then, let us choose $\mu = 1/(4S')$. For τ sufficiently small, we can now apply Lemma 3.5 and easily conclude a bound for $\{\overline{U}_\tau\}$ in $L^\infty(0,T)$. Hence,

$$\|\overline{U}_\tau\|_{L^\infty(0,T;H)} + \|\underline{U}_\tau\|_{L^\infty(0,T;H)} + \|U_\tau\|_{L^\infty(0,T;H)} \le C, \tag{3.34}$$

for a constant C independent of τ.

Turning back to (3.33) and adding it up for $k = 1, \ldots, n$, we obtain

$$\int_0^{t_n} \|U_\tau'(s)\|_H^2 ds + \phi(\overline{U}_\tau(t_n)) \le \phi(u_0) + M^2 T + M^2 \int_0^{t_n} \|\underline{U}_\tau(s)\|_H^2 ds, \tag{3.35}$$

whence there exists a positive constant C, independent of t and τ, such that

$$\phi(\overline{U}_\tau(t)) \le C, \quad \text{and} \quad \phi(U_\tau(t)) \le C \tag{3.36}$$

by convexity.

Moreover, thanks to (3.7) and the estimate (3.34), we have that $\phi(\overline{U}_\tau)$ is bounded in $L^\infty(0,T)$, so that the *energy estimate* (3.35) also gives

$$\{U_\tau\} \quad \text{is bounded in } H^1(0,T;H). \tag{3.37}$$

Recall that $\overline{W}_\tau = \overline{\alpha}_\tau \overline{\xi}_\tau$, for some $\overline{\xi}_\tau(t) \in B_H(U_\tau'(t))$ for a.e. $t \in (0,T)$. Hence, thanks to (3.8) and (3.37), we have that the sequence $\{\overline{\xi}_\tau\}$ is bounded in $L^2(0,T;H)$. Then, by (3.1), we deduce that

$$\{\overline{W}_\tau\} \quad \text{is bounded in } L^2(0,T;H). \tag{3.38}$$

Furthermore, by a comparison in (3.24) and (3.6), we also have that

$$\{\overline{V}_\tau\} \quad \text{is bounded in } L^2(0,T;H). \tag{3.39}$$

Finally, observe that

$$\|\overline{U}_\tau - U_\tau\|_{L^\infty(0,T;H)} = C\tau^{1/2}, \tag{3.40}$$

(an analogous estimate holds for $\underline{U}_\tau$), as a consequence of

$$\|u_\tau^n - U_\tau(t)\|_H^2 \le \tau \int_{t_{n-1}}^{t_n} \|U_\tau'(s)\|_H^2 \, ds \le C\tau, \quad t_{n-1} \le t \le t_n, \ n = 1, \ldots, N_\tau.$$

Compactness of the approximate solutions. *For any sequence* $\{\tau_k\}$ *of time steps such that* $\tau_k \downarrow 0$ *as* $k \uparrow \infty$*, we can find a further subsequence (still labelled* τ_k*), a limit function* $u \in H^1(0,T;H)$*, and* $w, v \in L^2(0,T;H)$*, such that as* $k \uparrow +\infty$*,*

$$\overline{U}_{\tau_k}, \underline{U}_{\tau_k}, U_{\tau_k}, \to u \quad \text{in } L^\infty(0,T;H), \tag{3.41}$$

$$U_{\tau_k}' \rightharpoonup u' \quad \text{in } L^2(0,T;H), \tag{3.42}$$

$$\overline{W}_{\tau_k} \rightharpoonup w \quad \text{and} \quad \overline{V}_{\tau_k} \rightharpoonup v \quad \text{in } L^2(0,T;H). \tag{3.43}$$

Indeed, the estimate (3.37) and the inequality

$$\|U_\tau(t) - U_\tau(s)\|_H \le (t-s)^{\frac{1}{2}} \|U'{}_\tau\|_{L^2(0,T;H)},$$

ensure that $\{U_\tau\}$ is *equicontinuous* on H for τ sufficiently small. On the other hand, thanks to (3.34) and (3.36), we may conclude that $\{U_\tau(t)\}_\tau$ is contained in some sublevel of the function $u \mapsto \phi(u) + S\|u\|_H^2$. Hence, by (3.4), the sequence $\{U_\tau(t)\}_\tau$ is relatively compact in H for every $t \in [0,T]$. Thanks to the equicontinuity property, the Ascoli compactness Theorem yields that $\{U_\tau\}_\tau$ is relatively compact in $C^0([0,T];H)$.

Hence, (3.41) follows as well, thanks to (3.40). Moreover, (3.42) and (3.43) follow from (3.37) and (3.38)–(3.39) by standard weak compactness results.

Passage to the limit and conclusion of the proof. As a consequence of (3.41), of (3.5)–(3.6) and of the Lebesgue theorem, we also have for all $1 \leq p < \infty$,

$$\mathcal{F}(\underline{U}_{\tau_k}) \to \mathcal{F}(u) \quad \text{in } L^p(0,T;H) \ \text{ as } k \uparrow \infty. \tag{3.44}$$

Then, also taking into account (3.42)–(3.43), we manage to pass to the limit in (3.24) and conclude that the triplet (u, w, ξ) fulfills (3.13). Moreover, (3.12) follows from (3.41), (3.43), and the strong-weak closedness of (the maximal monotone operator realizing) $\partial\phi$ in $L^2(0,T;H)$.

It remains to check (3.11): to this aim, for all $\tau > 0$ we consider the operator $\overline{\mathscr{B}}_\tau$ realizing the family of the operators $\{\overline{\mathcal{B}}_\tau(t)\}$ in $L^2(0,T;H)$ (see Lemma 3.4). Thanks to (3.1), (3.22), and Lemma 3.4, we have that

$$\overline{\mathscr{B}}_{\tau_k} \quad G\text{-converges to} \quad \mathscr{B} \quad \text{in } L^2(0,T;H) \text{ as } k \uparrow \infty, \tag{3.45}$$

$\mathscr{B}$ being the realization of the family of operators $\{\mathcal{B}(t)\}$ associated with the function α. Hence, in view of (3.25), (3.42), (3.43), and the closedness property (A.2) of G-convergence, (3.11) follows if we prove that

$$\limsup_{k\uparrow\infty} \int_0^T \left(\overline{W}_{\tau_k}(t), U'_{\tau_k}(t)\right)_H dt \leq \int_0^T (w(t), u'(t))_H dt. \tag{3.46}$$

Thus, we test (3.24) by U'_{τ_k} and integrate on the interval $(0,T)$. This leads to

$$\int_0^T \left(\overline{W}_{\tau_k}(t), U'_{\tau_k}(t)\right)_H dt = -\int_0^T \|U'_{\tau_k}(t)\|_H^2 dt - \int_0^T \left(\overline{V}_{\tau_k}(t), U'_{\tau_k}(t)\right)_H dt + \int_0^T \left(\mathcal{F}(\underline{U}_{\tau_k}(t)), U'_{\tau_k}(t)\right)_H dt.$$

Therefore, taking the $\limsup_{k\uparrow\infty}$ of both sides we obtain

$$\limsup_{k\uparrow\infty} \int_0^T \left(\overline{W}_{\tau_k}(t), U'_{\tau_k}(t)\right)_H dt \leq -\liminf_{k\uparrow\infty} \int_0^T \|U'_{\tau_k}(t)\|_H^2 dt + \lim_{k\uparrow\infty} \int_0^T \left(\mathcal{F}(\underline{U}_{\tau_k}(t)), U'_{\tau_k}(t)\right)_H dt - \liminf_{k\uparrow\infty} \sum_{j=1}^{N_\tau} (v_\tau^j, u_\tau^j - u_\tau^{j-1})_H.$$

The first and the second term on the right-hand side of the above inequality can be easily dealt with in view of the convergences (3.42) and (3.44). As for the third summand, due to (3.32) it is bounded from above by

$$\phi(u_0) - \liminf_{k\uparrow\infty} \phi(\overline{U}_{\tau_k}(T)) = \phi(u_0) - \liminf_{k\uparrow\infty} \phi(U_{\tau_k}(T)) \leq \phi(u_0) - \phi(u(T)),$$

where we have used that, by construction, $\overline{U}_{\tau_k}$ and U_{τ_k} coincide on the nodes of the partition $\mathcal{P}_{\tau_k}$, the uniform convergence (3.41), and the lower semicontinuity of ϕ.

Hence, (3.46) follows from

$$\begin{aligned}&\limsup_{k\uparrow\infty}\int_0^T \left(\overline{W}_{\tau_k}(t), {U'}_{\tau_k}(t)\right)_H dt\\ &\le \phi(u(0)) - \phi(u(T)) - \int_0^T \|u'(t)\|_H^2 \, dt + \int_0^T (\mathcal{F}(u(t)), u'(t))_H \, dt\\ &= \int_0^T (-v(t) - u'(t) + \mathcal{F}(u(t)), u'(t))_H \, dt = \int_0^T (w(t), u'(t))_H dt,\end{aligned}$$

where we have employed the chain rule [8, Lemma 3.3, p. 73] for $\partial\phi$. □

4. Existence for Problem 2.2.

Throughout this section, we will set $\varepsilon = 1$ in (2.15).

4.1. *An approximate problem.* Let $\{\beta_\nu\}_{\nu>0}$ be the sequence of the Yosida regularizations of β (see e.g. [8, Chap. II]): standard results in the theory of maximal monotone operators ensure that $\beta_\nu \in C^{\mathrm{Lip}}(\mathbb{R})$, with Lipschitz constant $1/\nu$. We also recall that, for every $\nu > 0$, β_ν is the derivative of the Moreau-Yosida approximation $\widehat{\beta_\nu}$ of $\widehat{\beta}$; in view of (2.7), for every $\nu > 0$ $\widehat{\beta_\nu}(r) \ge \widehat{\beta}(r) \ge 0$ for all $r \in \mathbb{R}$.

We approximate Problem 2.2 by the following.

Problem 4.1 (**Problem $\mathbf{P}_\nu$**). Find $\vartheta_\nu \in H^1(0,T;V') \cap C^0([0,T];H) \cap L^2(0,T;V)$, and $\chi_\nu \in H^1(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;W)$, fulfilling (2.14), (2.16), and

$$\partial_t\chi - \eta(\vartheta, \nabla\chi)(\partial_t\chi)^- + A\chi + \beta_\nu(\chi) + \sigma'(\chi) = \vartheta \quad \text{in } H, \quad \text{a.e. in } (0,T). \tag{4.1}$$

In the sequel, we first establish an existence result for Problem $\mathbf{P}_\nu$, and then we show that any sequence $\{(\vartheta_\nu, \chi_\nu)\}$ of solutions to Problem $\mathbf{P}_\nu$ converges, up to a subsequence, to a pair (ϑ, χ) solving Problem 2.2.

Proposition 4.2. Assume (2.1)–(2.2), (2.4)–(2.6), and (2.9)–(2.10). Then, for any $\nu > 0$ Problem $\mathbf{P}_\nu$ admits a solution $(\vartheta_\nu, \chi_\nu)$.

We are going to prove Proposition 4.2 by applying the Schauder fixed point theorem to a suitably defined *solution operator.*

Solution operator for the approximate problem. Preliminarily, we need the following result.

Lemma 4.3. Under the assumptions (2.1)–(2.2), and (2.4)–(2.6), for any $\chi^0 \in V$, $h \in L^2(0,T;H)$ and $j \in L^2(0,T;V)$ there exists a unique $\chi \in H^1(0,T;H) \cap C^0([0,T;V]) \cap L^2(0,T;W)$ solving the Cauchy problem

$$\begin{gathered}\partial_t\chi - \eta(h, \nabla j)(\partial_t\chi)^- + A\chi + \beta_\nu(\chi) + \sigma'(\chi) = h \quad \text{in } H, \quad \text{a.e. in } (0,T)\\ \chi(0) = \chi^0.\end{gathered} \tag{4.2}$$

Moreover, there exists a constant $C \ge 0$, only depending on T, $|\Omega|$, ν, and Λ_σ, such that for any $t \in (0,T]$,

$$\|\chi\|_{H^1(0,t;H)\cap C^0([0,t];V)\cap L^2(0,t;W)} \le C\left(\|\chi^0\|_V + \|h\|_{L^2(0,t;H)}\right). \tag{4.3}$$

Proof. Note that (4.2) may be recast in the abstract form (3.13) by setting

$$B_H : H \to H \quad \text{induced by} \quad B : \mathbb{R} \to \mathbb{R} \quad \text{with} \quad B(s) := -(s)^- \quad \forall\, s \in \mathbb{R},$$

$$\alpha : \Omega \times (0,T) \to \mathbb{R} \quad \text{given by} \quad \alpha(x,t) := \eta(h(x,t), \nabla j(x,t)) \quad \text{for a.e. } (x,t) \in Q,$$

$$\phi : H \to [0,+\infty) \quad \phi(v) := \begin{cases} \int_\Omega \frac{1}{2}|\nabla v|^2 & \text{if } v \in H^1(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

$$\mathcal{F} : H \to H \quad \text{defined by} \quad \mathcal{F}(v) := -\beta_\nu(v) - \sigma'(v) \quad \forall v \in H,$$

$$f(t) := h(t) \quad \text{for a.e. } t \in (0,T).$$

Indeed, it is easy to check that, in the framework of (2.1)–(2.2) and (2.4)–(2.6), the above choices fulfill the assumptions of Theorem 3.2. Let us only note that, since β_ν and σ' are Lipschitz continuous on $\mathbb{R}$, for all $v \in H$, $\beta_\nu(v) + \sigma'(v) \in H$, and the boundedness condition (3.6) easily follows.

Hence, we may conclude that there exists a solution $\chi \in H^1(0,T;H)$ to the Cauchy problem (4.2). Further, testing the equation by $\partial_t \chi$ and integrating on the interval $(0,t)$, we obtain

$$\begin{aligned} &\int_0^t \|\partial_t\chi(s)\|_H^2 ds + \int_0^t \int_\Omega \eta(h(x,s), \nabla j(x,s)) |(\partial_t \chi(x,s))^-|^2 dx ds \\ &\quad + \frac{1}{2}\|\nabla\chi(t)\|_H^2 + \int_\Omega \widehat{\beta_\nu}(\chi(x,t))dx \le \frac{1}{2}\|\nabla\chi^0\|_H^2 dx + \int_\Omega \widehat{\beta}_\nu(\chi^0(x)) \qquad (4.4) \\ &+ C\|\chi^0\|_H^2 + \int_0^t \|h(s)\|_H^2 ds + \Lambda_\sigma^2 \int_0^t \|\chi(s) - \chi^0\|_H^2 ds + \frac{3}{4}\int_0^t \|\partial_t\chi(s)\|_H^2 ds, \end{aligned}$$

where we have used the elementary inequality

$$\int_0^t (h(s), \partial_t\chi(s))_H \le \int_0^t \|h(s)\|_H^2 ds + \frac{1}{4}\int_0^t \|\partial_t\chi(s)\|_H^2 ds$$

and exploited (2.6) to conclude that

$$\begin{aligned} &\int_0^t (\sigma'(\chi(s)), \partial_t\chi(s))_H \, ds \\ &\le \int_0^t \|\sigma'(\chi(s)) - \sigma'(\chi^0)\|_H^2 ds + T\|\sigma'(\chi^0)\|_H^2 + \frac{1}{2}\int_0^t \|\partial_t\chi(s)\|_H^2 ds \\ &\le T\|\sigma'(\chi^0)\|_H^2 + \Lambda_\sigma^2 \int_0^t \|\chi(s) - \chi^0\|_H^2 ds + \frac{1}{2}\int_0^t \|\partial_t\chi(s)\|_H^2 ds. \end{aligned}$$

Note that the second integral term on the left-hand side of (4.4) is nonnegative, as well as the fourth term. Hence, it is not difficult to check that there exists a positive constant C, depending on T, $|\Omega|$, and Λ_σ, such that

$$\frac{1}{4}\int_0^t \|\partial_t\chi(s)\|_H^2 ds \le C\left(\|\chi^0\|_V^2 + \|h\|_{L^2(0,t;H)}^2 + \int_0^t \left(\int_0^s \|\partial_t\chi(r)\|_H^2 dr \right) ds \right).$$

Thus, the Gronwall Lemma yields an a priori estimate for $\|\chi\|_{H^1(0,t;H)}$ in terms of $\|\chi^0\|_V$ and of $\|h\|_{L^2(0,t;H)}$. On account of (4.4), we also deduce an estimate for $\|\nabla\chi\|_{L^\infty(0,t;H)}$, hence for $\|\chi\|_{L^\infty(0,t;V)}$. Furthermore, note that

$$\|\sqrt{\eta(h,\nabla j)}(\partial_t\chi)^-\|_{L^2(0,t;H)} + \|\sigma'(\chi)\|_{L^\infty(0,t;H)} \le C\left(\|h\|_{L^2(0,t;H)} + \|\chi^0\|_V\right),$$

while

$$\|\beta_\nu(\chi)\|_{L^\infty(0,t;H)} \leq C_\nu \left(\|h\|^2_{L^2(0,t;H)} + \|\chi^0\|^2_V\right)$$

(the constant C_ν in fact depends on ν as well, and blows up as $\nu \downarrow 0$). By comparison in (4.2), we obtain $\|A\chi\|_{L^2(0,t;H)} \leq C(1 + \|\chi^0\|_V)$, hence the estimate for $\|\chi\|_{L^2(0,t;W)}$ follows from standard elliptic regularity results. It is also well known that $H^1(0,T;H) \cap L^2(0,T;W)$ is continuously embedded in $C^0([0,T];V)$, whence $\chi \in C^0([0,T];V)$.

In order to prove uniqueness (the same argument would also yield a result of continuous dependence on the data χ_0 and h), let $\chi_1,\ \chi_2 \in H^1(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;W)$ be two solutions to (4.2), and let us denote by $\widetilde{\chi}$ their difference $\chi_1 - \chi_2$. Thus, $\widetilde{\chi}$ satisfies

$$\begin{aligned}&\partial_t\widetilde{\chi}(t) - \eta(h(t),\nabla j(t))(\partial_t\chi_1(t))^- + \eta(h(t),\nabla j(t))(\partial_t\chi_2(t))^- + A\widetilde{\chi}(t)\\ &\quad + \beta_\nu(\chi_1(t)) - \beta_\nu(\chi_2(t)) + \sigma'(\chi_1(t)) - \sigma'(\chi_2(t)) = 0 \quad \text{in } H \quad \text{for a.e. } t \in (0,T),\end{aligned}$$

which we test by $\partial_t\widetilde{\chi}$. Upon integrating on $(0,t)$, $0 < t \leq T$, we obtain

$$\begin{aligned}&\int_0^t \|\partial_t\widetilde{\chi}(s)\|^2_H\,ds + \frac{1}{2}\|\nabla\widetilde{\chi}(t)\|^2_H\\ &+ \int_0^t \left(-\eta(h(s),\nabla j(s))(\partial_t\chi_1(s))^- + \eta(h(s),\nabla j(s))(\partial_t\chi_2(s))^-, \partial_t\widetilde{\chi}(s)\right)_H ds\\ &= \int_0^t (\beta_\nu(\chi_1(s)) + \sigma'(\chi_1(s)) - \beta_\nu(\chi_2(s)) - \sigma'(\chi_2(s)), \partial_t\widetilde{\chi}(s))_H\,ds\\ &\leq \frac{1}{2}\int_0^t \|\partial_t\widetilde{\chi}(s)\|^2_H\,ds + \left(\frac{1}{\nu^2} + \Lambda^2_\sigma\right)\int_0^t \|\widetilde{\chi}(s)\|^2_H\,ds.\end{aligned}$$

By monotonicity, we have that

$$\int_0^t \left(-\eta(h(s),\nabla j(s))(\partial_t\chi_1(s))^- + \eta(h(s),\nabla j(s))(\partial_t\chi_2(s))^-, \partial_t\widetilde{\chi}(s)\right)_H ds \geq 0,$$

hence we deduce that

$$\frac{1}{2}\int_0^t \|\partial_t\widetilde{\chi}(s)\|^2_H\,ds \leq \left(\frac{1}{\nu^2} + \Lambda^2_\sigma\right) T \int_0^t \left(\int_0^s \|\partial_t\widetilde{\chi}(r)\|^2_H\,dr\right) ds,$$

which yields $\widetilde{\chi}(t) = 0$ for all $t \in [0,T]$, again by the Gronwall lemma. □

Let $(\overline{\vartheta},\overline{\chi}) \in L^2(0,T;H) \times L^2(0,T;V)$ be given: Lemma 4.3 applies, yielding the existence of a unique $\widehat{\chi}$ fulfilling

$$\begin{cases}\widehat{\chi} \in H^1(0,T;H) \cap C^0([0,T;V]) \cap L^2(0,T;W) \quad \text{with} \quad \widehat{\chi}(0) = \chi_0 \quad \text{and}\\ \partial_t\widehat{\chi} - \eta(\overline{\vartheta}(t),\nabla\overline{\chi}(t))\,(\partial_t\widehat{\chi})^- + A\chi + \beta_\nu(\widehat{\chi}) + \sigma'(\widehat{\chi}) = \overline{\vartheta}(t) \quad \text{in } H,\\ \qquad \text{for a.e. } t \in (0,T).\end{cases} \tag{4.5}$$

On the other hand, easily adapting a standard result in the theory of parabolic equations (see [16, Thm. 4.1, p. 238]), or applying the theory of nonlinear semigroups generated by maximal monotone operators (cf. [7, Thm. 2.1, p. 189] or [8, Thm. 3.6, p. 72]), we conclude that there exists a unique

$$\begin{cases}\widehat{\vartheta} \in H^1(0,T;V') \cap C^0([0,T];H) \cap L^2(0,T;V) \quad \text{with} \quad \widehat{\vartheta}(0) = \vartheta_0 \quad \text{and}\\ \partial_t\widehat{\vartheta} + J\widehat{\vartheta} = F - \partial_t\widehat{\chi} \quad \text{in } V', \quad \text{a.e. in } (0,T).\end{cases} \tag{4.6}$$

On account of (4.5) and (4.6), we define the *solution* operator $\mathcal{S} : L^2(0,T;H) \times L^2(0,T;V) \to L^2(0,T;H) \times L^2(0,T;V)$ by

$$\mathcal{S}(\overline{\vartheta}, \overline{\chi}) := (\widehat{\vartheta}, \widehat{\chi}). \tag{4.7}$$

Henceforth, we will use the simpler notation (ϑ, χ) for $(\widehat{\vartheta}, \widehat{\chi})$. Of course, any fixed point $(\overline{\vartheta}, \overline{\chi})$ for $\mathcal{S}$ yields a solution to Problem $\mathbf{P}_\nu$.

4.2. *Proof of Proposition* 4.2. Given $R_0 > 0$ and a final time $T_0 > 0$ (which will be specified later), we set

$$\mathcal{Y} := \{(w,u) \in L^2(0,T_0;H) \times L^2(0,T_0;V) \ : \ \max\{\|w\|_{L^2(0,T_0;H)}, \|u\|_{L^2(0,T_0;V)}\} \le R_0\}.$$

Proposition 4.4. Assume (2.1)–(2.2), (2.4)–(2.6), and (2.9)–(2.10). Then, for any $R_0 > 0$ there exists $T_0 \in (0,T]$ such that

$$\mathcal{S} \text{ maps } \mathcal{Y} \text{ into itself;} \tag{4.8}$$

$$\mathcal{S} : \mathcal{Y} \to \mathcal{Y} \quad \text{is a continuous operator;} \tag{4.9}$$

$$\mathcal{S} : \mathcal{Y} \to \mathcal{Y} \quad \text{is a compact operator.} \tag{4.10}$$

Proof. **Ad** (4.8)**.** Fix $(\overline{\vartheta}, \overline{\chi}) \in \mathcal{Y}$, and let $(\vartheta, \chi) := \mathcal{S}(\overline{\vartheta}, \overline{\chi})$. It follows from Lemma 4.3 (cf. estimate (4.3)), that there exists a constant C, only depending on T, $|\Omega|$ and Λ_σ, such that

$$\|\chi\|_{H^1(0,T_0;H)\cap C^0([0,T_0];V)} \le C\left(\|\chi_0\|_V + \|\overline{\vartheta}\|_{L^2(0,T_0;H)}\right) \le C\left(\|\chi_0\|_V + R_0\right). \tag{4.11}$$

On the other hand, by construction the pair (ϑ, χ) in particular fulfills problems (4.5)–(4.6) on the interval $(0,T_0)$. Let us test (4.6) by ϑ and integrate on $(0,t)$, $t \in (0,T_0]$. Also taking into account (2.12), we obtain

$$\begin{aligned}\frac{1}{2}\|\vartheta(t)\|_H^2 + \int_0^t \|\vartheta(s)\|_V^2 ds \le \frac{1}{2}\|\vartheta_0\|_H^2 + \frac{1}{2}\int_0^t \|F(s)\|_{V'}^2 ds + \frac{1}{2}\int_0^t \|\vartheta(s)\|_V^2 ds \\ + \int_0^t \|\partial_t \chi(s)\|_H^2 ds + \frac{1}{4}\int_0^t \|\vartheta(s)\|_H^2 ds,\end{aligned} \tag{4.12}$$

whence

$$\|\vartheta(t)\|_H^2 + \int_0^t \|\vartheta(s)\|_V^2 ds \le C\left(\|\vartheta_0\|_H^2 + \int_0^t \|F(s)\|_{V'}^2 ds + \|\chi_0\|_V^2 + R_0^2\right). \tag{4.13}$$

Therefore, there exists a constant $\mathcal{C}$, only depending on T, $|\Omega|$, R_0, $\|F\|_{L^2(0,T;V')}$, $\|\vartheta_0\|_H$, and $\|\chi_0\|_V$, such that

$$\max\left\{\|\chi\|_{L^2(0,T_0;V)}, \|\vartheta\|_{L^2(0,T_0;H)}\right\} \le \mathcal{C} T_0^{1/2}. \tag{4.14}$$

Choosing $0 < T_0^{1/2} \le R_0/\mathcal{C}$, we conclude that $\mathcal{S}(\overline{\vartheta}, \overline{\chi}) \in \mathcal{Y}$, whence (4.8).

Ad (4.10)**.** In fact, for any $(\overline{\vartheta}, \overline{\chi}) \in \mathcal{Y}$ we have the following additional estimates for the pair $(\vartheta, \chi) = \mathcal{S}(\overline{\vartheta}, \overline{\chi})$:

$$\begin{gathered}\|\chi\|_{L^2(0,T_0;W)} \le C\left(\|\chi_0\|_V + \|\overline{\vartheta}\|_{L^2(0,T_0;H)}\right) \le C', \\ \|\vartheta\|_{H^1(0,T_0;V')\cap C^0([0,T];H)\cap L^2(0,T_0;V)} \le C'\end{gathered} \tag{4.15}$$

where the constant C' only depends on T, $|\Omega|$, R_0, ν, $\|F\|_{L^2(0,T;V')}$, $\|\vartheta_0\|_H$, and $\|\chi_0\|_V$, but *not* on $(\overline{\vartheta}, \overline{\chi})$. Indeed, the estimate for $\|\chi\|_{L^2(0,T_0;W)}$ is a consequence of (4.3). The bound for ϑ follows from (4.13) and from arguing by comparison in (4.6). Thanks to the a priori estimates (4.11) and (4.15) and recalling the compactness results [19, Thm. 5, Cor. 4], we conclude that $\mathcal{S}$ is a compact operator.

Ad (4.9)**.** Let $\{(\overline{\vartheta}_n, \overline{\chi}_n)\}_n \subset \mathcal{Y}$ fulfill

$$\overline{\vartheta}_n \to \overline{\vartheta}_\infty \quad \text{in } L^2(0,T_0;H) \quad \text{and} \quad \overline{\chi}_n \to \overline{\chi}_\infty \text{ in } L^2(0,T_0;V) \tag{4.16}$$

as $n \uparrow \infty$. Up to a subsequence, we may assume that for a.e. $(x,t) \in \Omega \times (0,T_0)$, $\overline{\vartheta}_n(x,t) \to \overline{\vartheta}_\infty(x,t)$ and $\nabla\overline{\chi}_n(x,t) \to \nabla\overline{\chi}_\infty(x,t)$. Hence, by (2.1), (2.2) and the Lebesgue theorem, we end up with

$$\eta(\overline{\vartheta}_n, \nabla\overline{\chi}_n) \to \eta(\overline{\vartheta}_\infty, \nabla\overline{\chi}_\infty) \quad \text{in } L^2(0,T_0;H). \tag{4.17}$$

Now, the estimates (4.11), (4.14), (4.15) for the corresponding sequence $\mathcal{S}(\overline{\vartheta}_n, \overline{\chi}_n) =: (\vartheta_n, \chi_n)$ yield

$$\|\chi_n\|_{H^1(0,T_0;H)\cap C^0([0,T_0];V)\cap L^2(0,T_0;W)} + \|\vartheta_n\|_{H^1(0,T_0;V')\cap C^0([0,T_0];H)\cap L^2(0,T_0;V)} \le C,$$

independently of $n \in \mathbb{N}$. Standard weak compactness results, as well as the aforementioned [19, Thm. 5, Cor. 4], guarantee that there exists a subsequence $\{n_k\}_k$, and a limit pair (χ, ϑ), with $\chi \in H^1(0,T_0;H) \cap C^0([0,T_0];V) \cap L^2(0,T_0;W)$, and $\vartheta \in H^1(0,T_0;V') \cap C^0([0,T_0];H) \cap L^2(0,T_0;V)$, such that the following convergences hold for $\{\chi_{n_k}\}$ and $\{\vartheta_{n_k}\}$ as $k \uparrow \infty$:

$$\chi_{n_k} \rightharpoonup^* \chi \quad \text{in } H^1(0,T_0;H) \cap L^\infty(0,T_0;V) \cap L^2(0,T_0;W); \tag{4.18}$$

$$\chi_{n_k} \to \chi \quad \text{in } C^0([0,T_0];H) \cap L^p(0,T_0;V) \ \text{ for any } 1 \le p < \infty; \tag{4.19}$$

$$\vartheta_{n_k} \rightharpoonup^* \vartheta \quad \text{in } H^1(0,T_0;V') \cap L^\infty(0,T_0;H) \cap L^2(0,T_0;V); \tag{4.20}$$

$$\vartheta_{n_k} \to \vartheta \quad \text{in } C^0([0,T_0];V') \cap L^p(0,T_0;H) \ \text{ for any } 1 \le p < \infty. \tag{4.21}$$

By the Lipschitz continuity of β_ν and σ', we readily deduce from (4.19) that $\beta_\nu(\chi_{n_k}) \to \beta_\nu(\chi)$ and $\sigma'(\chi_{n_k}) \to \sigma'(\chi)$ in $L^p(0,T_0;H)$ for any $1 \le p < \infty$. Moreover, there exists $\zeta \in L^2(0,T_0;H)$ such that

$$-\eta(\overline{\vartheta}_{n_k}, \nabla\overline{\chi}_{n_k})(\partial_t \chi_{n_k})^- \rightharpoonup \zeta \quad \text{in } L^2(0,T_0;H). \tag{4.22}$$

By (4.16) and the convergences (4.18)–(4.22) so far retrieved, we are able to pass to the limit in the equations (4.5) and (4.6) fulfilled by χ_{n_k} and ϑ_{n_k}. Thus, we find

$$\partial_t\vartheta + \partial_t\chi + J\vartheta = F \quad \text{in } V' \quad \text{a.e. in } (0,T_0); \tag{4.23}$$

$$\partial_t\chi + \zeta + A\chi + \beta_\nu(\chi) + \sigma'(\chi) = \overline{\vartheta}_\infty \quad \text{in } H \quad \text{a.e. in } (0,T_0). \tag{4.24}$$

Actually, we have

$$\zeta(x,t) = -\eta(\overline{\vartheta}_\infty(x,t), \overline{\chi}_\infty(x,t))(\partial_t\chi(x,t))^- \quad \text{for a.e. } (x,t) \in \Omega \times (0,T). \tag{4.25}$$

By (4.17) and Lemma 3.4, the maximal monotone operator $\mathscr{B}_n : L^2(0,T_0;H) \to L^2(0,T_0;H)$, defined for all $n \in \mathbb{N}$ and for all $v \in L^2(0,T_0;H)$ by

$$\mathscr{B}_n(v) := -\eta(\overline{\vartheta}_n(x,t), \nabla\overline{\chi}_n(x,t))(v(x,t))^- \quad \text{for a.e. } (x,t) \in \Omega \times (0,T_0), \tag{4.26}$$

converges in the sense of graphs to the operator $\mathscr{B}_\infty$, still defined by formula (4.26) with $\eta(\overline{\vartheta}_\infty, \nabla\overline{\chi}_\infty)$ instead of $\eta(\overline{\vartheta}_n, \nabla\overline{\chi}_n)$. Thus, in view of (A.2) (see Section A), (4.25) follows by noting that

$$\begin{aligned}
&\limsup_{k\uparrow\infty} \int_0^{T_0} \int_\Omega (-\eta(\overline{\vartheta}_{n_k}(x,t), \nabla\overline{\chi}_{n_k}(x,t))(\partial_t\chi_{n_k})^-(x,t)\partial_t\chi_{n_k}(x,t)\; dxdt \\
&\leq \limsup_{k\uparrow\infty} \left(-\frac{1}{2}\|\nabla\chi_{n_k}(T_0)\|_H^2 + \frac{1}{2}\|\nabla\chi_0\|_H^2 \right) \\
&- \liminf_{k\uparrow\infty} \int_0^{T_0} \left(\|\partial_t\chi_{n_k}(t)\|_H^2 + (\beta_\nu(\chi_{n_k}(t)) + \sigma'(\chi_{n_k}(t)), \partial_t\chi_{n_k}(t))_H \right) dt \\
&+ \lim_{k\uparrow\infty} \int_0^{T_0} (\overline{\vartheta}_{n_k}(t), \partial_t\chi_{n_k}(t))_H dt \leq -\frac{1}{2}\|\nabla\chi(T_0)\|_H^2 + \frac{1}{2}\|\nabla\chi_0\|_H^2 \\
&- \int_0^{T_0} \left(\|\partial_t\chi(t)\|_H^2 + (\beta_\nu(\chi(t)) + \sigma'(\chi(t)), \partial_t\chi(t))_H - (\overline{\vartheta}_\infty(t), \partial_t\chi(t))_H \right) dt \\
&= \int_0^{T_0} \int_\Omega \zeta(x,t)\partial_t\chi(x,t)\; dxdt.
\end{aligned} \tag{4.27}$$

Observe that the first inequality in the chain above follows by testing (4.5) (written for χ_{n_k}) by $\partial_t\chi_{n_k}$, and the second one by combining the strong and weak convergences (4.18)–(4.21) with (4.16); the final equality is due to (4.24).

Thanks to (4.23)–(4.24), and (4.25), we obtain that the limit pair (ϑ, χ) has the regularity required in (4.5)–(4.6), and fulfills

$$\partial_t\vartheta + \partial_t\chi + J\vartheta = F \quad \text{in } V' \quad \text{a.e. in } (0, T_0); \tag{4.28}$$

$$\partial_t\chi - \eta(\overline{\vartheta}_\infty, \nabla\overline{\chi}_\infty) + A\chi + \beta_\nu(\chi) + \sigma'(\chi) = \overline{\vartheta}_\infty \quad \text{in } H \quad \text{a.e. in } (0, T_0). \tag{4.29}$$

Hence, $(\vartheta, \chi) = \mathcal{S}(\overline{\vartheta}_\infty, \overline{\chi}_\infty)$ and, by uniqueness of the limit, the convergences (4.18)–(4.21) hold along the whole sequences $\{\vartheta_n\}$, $\{\chi_n\}$. In particular,

$$\mathcal{S}(\overline{\vartheta}_n, \overline{\chi}_n) \to \mathcal{S}(\overline{\vartheta}_\infty, \overline{\chi}_\infty) \quad \text{in } L^2(0, T_0; H) \times L^2(0, T_0; V),$$

which entails (4.9). □

Conclusion of the proof of Proposition 4.2. By the Schauder fixed point theorem, the solution operator $\mathcal{S} : \mathcal{Y} \to \mathcal{Y}$ has a fixed point (ϑ, χ), yielding by construction a *local* solution to Problem $\mathbf{P}_\nu$ on the time interval $[0, T_0]$.

Let us now perform the following estimates: first, we test (4.28) by ϑ, (4.29) by $\partial_t\chi$, add the resulting relations and integrate on $(0, t)$, $0 \leq t \leq T_0$. Upon cancellation of two terms, we easily obtain

$$
\begin{aligned}
&\frac{1}{2}\|\vartheta(t)\|_H^2 + \int_0^t \|\vartheta(s)_V^2 ds + \int_0^t \int_\Omega \eta(\vartheta(x,s), \nabla\chi(x,s))|(\partial_t\chi(x,s))^-|^2 \; dxds \qquad (4.30)\\
&\quad + \int_0^t \|\partial_t\chi(s)\|_H^2 \, ds + \frac{1}{2}\|\nabla\chi(t)\|_H^2 + \int_\Omega \widehat{\beta_\nu}(\chi(x,t))dx\\
&\le \frac{1}{2}\|\vartheta_0\|_H^2 + \int_0^t \langle F(s), \vartheta(s)\rangle \; ds + \frac{1}{2}\|\nabla\chi_0\|_H^2 + \int_\Omega \widehat{\beta_\nu}(\chi_0(x))dx\\
&\quad + \int_0^t (\sigma'(\chi(s)), \partial_t\chi(s))_H ds \le C(\|\chi_0\|_V^2 + \|\vartheta_0\|_H^2) + \frac{1}{2}\int_0^t \|F(s)\|_{V'}^2 \; ds\\
&\quad + \frac{1}{2}\int_0^t \|\vartheta(s)\|_V^2 \; ds + \frac{1}{2}\int_0^t \|\partial_t\chi(s)\|_H^2 ds + \frac{1}{2}\Lambda_\sigma^2 \int_0^t \|\chi(s) - \chi_0\|_H^2 ds,
\end{aligned}
$$

where we have used (2.12), and, in the last passage, the Lipschitz continuity of σ'. Arguing as in the proof of Lemma 4.3, using that $\widehat{\beta_\nu} \ge 0$ and applying Gronwall's Lemma, we deduce that

$$
\int_0^t \|\partial_t\chi(s)\|_H^2 ds \le C\left(\|\chi_0\|_V^2 + \|\vartheta_0\|_H^2 + \|F\|_{L^2(0,T;V')}^2\right)\exp(\Lambda_\sigma^2 T^2) \tag{4.31}
$$

for any $0 \le t \le T_0$. Hence, (4.30) and (4.31) yield that

$$
\begin{aligned}
\|\chi\|_{H^1(0,t;H)\cap C^0([0,t];V)} + \|\sqrt{\eta(\vartheta,\nabla\chi)}(\partial_t\chi)^-\|_{L^2(0,t;H)}&\\
+ \|\vartheta\|_{C^0([0,t];H)\cap L^2(0,t;V)} \le C&,
\end{aligned} \tag{4.32}
$$

for a constant C again depending only on $\|\chi_0\|_V$, $\|\vartheta_0\|_H$, and $\|F\|_{L^2(0,T;V')}$, but not on $t \in [0, T_0]$. A comparison argument in (4.28) and in (4.29) and standard elliptic regularity results entail the additional estimates

$$
\|\chi\|_{L^2(0,t;W)} + \|\vartheta\|_{H^1(0,t;V')} \le C. \tag{4.33}
$$

It is straightforward to realize that the *global estimates* (4.31)–(4.33) guarantee that the pair (ϑ, χ) can be extended to a solution of the system (4.5)–(4.6), on the *whole* interval $[0, T]$. □

4.3. *Passage to the limit in the approximate problem and conclusion of the proof of Theorem* 2.3. The proof of Theorem 2.3 follows from the following result, stating that any solution $(\vartheta_\nu, \chi_\nu)$ to Problem $\mathbf{P}_\nu$ converges to a solution (ϑ, χ) of Problem 2.2 as $\nu \downarrow 0$.

PROPOSITION 4.5. Assume (2.1)–(2.2), (2.4)–(2.6), and (2.9)–(2.10); let $\{(\vartheta_\nu, \chi_\nu)\}_\nu$ be the sequence of the solutions to $\mathbf{P}_\nu$. Then, there exists a subsequence $\nu_j \nearrow \infty$ for $j \uparrow \infty$, and a triplet (ϑ, χ, ξ), with $\vartheta \in H^1(0,T;V') \cap C^0([0,T];H) \cap L^2(0,T;V)$, $\chi \in H^1(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;W)$ and $\chi \in D(\widehat{\beta})$ a.e. in Q, and $\xi \in L^2(0,T;H)$, such that the convergences (4.18)–(4.21) hold for $\{\vartheta_{\nu_j}\}$, ϑ and $\{\chi_{\nu_j}\}$, χ as $j \uparrow \infty$, as well as

$$
\beta_{\nu_j}(\chi_{\nu_j}) \rightharpoonup \xi \quad \text{in } L^2(0,T;H) \quad \text{as } j \uparrow \infty. \tag{4.34}
$$

Moreover, $\xi \in \beta(\chi)$ a.e. in Q, and the triplet (ϑ, χ, ξ) is a solution to Problem 2.2.

Proof. Note that the a priori estimates (4.31) and (4.32) are indeed *independent* of the parameter ν, whence, also by a comparison in (2.14),

$$\begin{aligned}\|\chi_\nu\|_{H^1(0,T;H)\cap C^0([0,T];V)} + \|\sqrt{\eta(\vartheta_\nu,\nabla\chi_\nu)}(\partial_t\chi_\nu)^-\|_{L^2(0,T;H)} \\ + \|\vartheta_\nu\|_{H^1(0,T;V')\cap C^0([0,T];H)\cap L^2(0,T;V)} \leq C,\end{aligned} \tag{4.35}$$

for a constant C only depending on the data χ_0, ϑ_0 and F of the Problem. Hence, testing (2.15) by $\beta_\nu(\chi_\nu)$, and noting that

$$\int_0^t \langle A\chi_\nu(s), \beta_\nu(\chi_\nu(s))\rangle \, ds \geq 0$$

for all $t \in [0,T]$ by monotonicity, we readily deduce that

$$\|\beta_\nu(\chi_\nu)\|_{L^2(0,t;H)} + \|\chi_\nu\|_{L^2(0,t;W)} \leq C \quad \forall \nu > 0 \quad \forall t \in [0,T],$$

the second bound again by comparison in (2.15) and by elliptic regularity results.

By [19, Thm. 5, Cor. 4] and the aforementioned weak compactness results, there exists a subsequence $\{\nu_j\}$ and a quadruple $(\vartheta,\chi,\xi,\zeta)$ along which the convergences (4.18)–(4.21) and (4.34) hold, as well as

$$\zeta \in L^2(0,T;H), \quad -\eta(\vartheta_{\nu_j},\nabla\chi_{\nu_j})(\partial_t\chi_{\nu_j})^- \rightharpoonup \zeta \quad \text{in } L^2(0,T_0;H) \text{ as } j \uparrow \infty.$$

Note that the maximal monotone operator $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ induces a maximal monotone operator on $L^2(0,T;H)$. Thanks to [7, Prop. 1.1, p. 42], to conclude $\xi \in \beta(\chi)$ a.e. in Ω, it is sufficient to prove that

$$\limsup_{j\uparrow\infty} \int_0^T \int_\Omega \beta_{\nu_j}(\chi_{\nu_j}(x,t))\chi_{\nu_j}(x,t) \, dxdt \leq \int_0^T \int_\Omega \xi(x,t)\chi(x,t) \, dxdt,$$

which is a consequence of the strong convergence for χ_{ν_j} in $L^2(0,T;H)$ and of (4.34). Thus, passing to the limit in (2.14) and in (4.1), we find that the quadruple $(\vartheta,\chi,\xi,\zeta)$ fulfills (2.14) and

$$\partial_t\chi + \zeta + A\chi + \xi + \sigma'(\chi) = \vartheta, \quad \xi \in \beta(\chi), \quad \text{in } H \quad \text{for a.e. } t \in (0,T). \tag{4.36}$$

Hence, in order to conclude that (ϑ,χ,ξ) solves Problem 2.2, it remains to check

$$\zeta(x,t) = -\eta(\vartheta(x,t),\nabla\chi(x,t))(\partial_t\chi(x,t))^- \quad \text{for a.e. } (x,t) \in \Omega \times (0,T).$$

This can be verified by exactly repeating the argument for (4.25) in the proof of Proposition 4.4, i.e., by proving the analogue of the lim sup inequality (4.27). The computations for obtaining such inequality are the same as for Proposition 4.4, with the only exception of

$$\limsup_{j\uparrow\infty} \left(-\int_0^T \int_\Omega \beta_{\nu_j}(\chi_{\nu_j}(x,t))\partial_t\chi_{\nu_j}(x,t)dxdt \right) \leq -\int_0^T \int_\Omega \xi(x,t)\partial_t\chi(x,t)dxdt.$$

Indeed, the above inequality follows from

$$\begin{aligned}
&\liminf_{j\uparrow\infty}\int_0^T\int_\Omega \beta_{\nu_j}(\chi_{\nu_j}(x,t))\partial_t\chi_{\nu_j}(x,t)dxdt\\
&=\liminf_{j\uparrow\infty}\left(\int_\Omega \widehat{\beta_{\nu_j}}(\chi_{\nu_j}(x,t))dx-\int_\Omega \widehat{\beta_{\nu_j}}(\chi_0(x))dx\right)\\
&\geq \int_\Omega \left(\widehat{\beta}(\chi(x,t))-\widehat{\beta}(\chi_0(x))\right)dx=\int_0^T\int_\Omega \xi(x,t)\partial_t\chi(x,t)dx.
\end{aligned}$$

Here, we have applied the chain rule for convex l.s.c. functionals to get the first and the third identity. The intermediate inequality is a consequence of the fact that the integral functional on H associated with $\widehat{\beta_{\nu_j}}$ Mosco-converges (see Section A and (A.1)) to the integral functional on H associated with $\widehat{\beta}$, and of the strong convergence of $\chi_{\nu_j}(t)$ to $\chi(t)$ in H for all $t\in[0,T]$. ☐

5. Asymptotic analysis for Problem 2.2.

Proof of Theorem 2.7. The first part of our argument consists of finding proper a priori estimates on the sequences $\{\vartheta_\varepsilon\}$ and $\{\chi_\varepsilon\}$, in order to eventually apply suitable weak compactness results. We will often use the shorthand notation

$$\eta^\varepsilon \quad \text{for} \quad \eta(\vartheta_\varepsilon,\nabla\chi_\varepsilon).$$

First a priori estimate. We test (2.14) by ϑ_ε, (2.15) by $\partial_t\chi_\varepsilon$, add the resulting equations and integrate on $(0,t)$. Applying the chain rule [8, Lemma 3.3, p. 73] to the subdifferential β of the convex l.s.c. functional $\widehat{\beta}$, we obtain

$$\begin{aligned}
&\frac{1}{2}\|\vartheta_\varepsilon(t)\|_H^2+\int_0^t\|\vartheta_\varepsilon(s)\|_V^2ds+\varepsilon\int_0^t\|\partial_t\chi_\varepsilon(s)\|_H^2ds+\frac{1}{2}\|\nabla\chi_\varepsilon(t)\|_H^2\\
&+\int_0^t\int_\Omega \eta^\varepsilon(x,s)|(\partial_t\chi_\varepsilon(x,s))^-|^2dxds+\int_\Omega\left(\widehat{\beta}(\chi_\varepsilon(x,t))+\sigma(\chi_\varepsilon(x,t))\right)dx \qquad (5.1)\\
=&\frac{1}{2}\|\vartheta_0^\varepsilon\|_H^2+\frac{1}{2}\|\nabla\chi_0^\varepsilon\|_H^2+\int_\Omega\left(\widehat{\beta}(\chi_0^\varepsilon(x))+\sigma(\chi_0^\varepsilon(x))\right)dx+\int_0^t\langle F^\varepsilon(s),\vartheta_\varepsilon(s)\rangle\, ds.
\end{aligned}$$

Of course, the last term on the right-hand side of (5.1) is estimated in the obvious way:

$$\left|\int_0^t\langle F^\varepsilon(s),\vartheta_\varepsilon(s)\rangle\, ds\right|\leq\frac{1}{2}\int_0^t\|F^\varepsilon(s)\|_{V'}^2ds+\frac{1}{2}\int_0^t\|\vartheta_\varepsilon(s)\|_V^2ds.$$

Moreover, by (2.6), there exists a positive constant C, also depending on Λ_σ, such that

$$\int_\Omega\sigma(\chi_0^\varepsilon(x))\leq C\left(\|\chi_0^\varepsilon\|_H^2+1\right).$$

Taking into account (2.19), and that by (2.17) the sequences $\{\vartheta_0^\varepsilon\}$, $\{\chi_0^\varepsilon\}$, and $\widehat{\beta}(\chi_0^\varepsilon)$ are bounded in H, in V, and in $L^1(\Omega)$ respectively, we conclude that

$$\int_\Omega\left(\widehat{\beta}(\chi_\varepsilon(x,t))+\sigma(\chi_\varepsilon(x,t))\right)dx\leq C$$

for a positive constant C independent of ε, whence we infer an a priori bound for χ_ε in $L^\infty(0,T;H)$ in view of (2.8). In the end, (5.1) yields that there exists a constant $C>0$

such that

$$\|\vartheta_\varepsilon\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|\chi_\varepsilon\|_{L^\infty(0,T;V)} + \varepsilon^{1/2}\|\partial_t\chi_\varepsilon\|_{L^2(0,T;H)} \le C \quad \forall \varepsilon > 0. \tag{5.2}$$

Second a priori estimate. Furthermore, it follows from the previous estimate that

$$\|\sqrt{\eta(\vartheta_\varepsilon, \nabla\chi_\varepsilon)}(\partial_t\chi_\varepsilon)^-\|_{L^2(0,T;H)} \le C \quad \forall\varepsilon > 0, \tag{5.3}$$

whence, for a.e. $t \in (0,T)$,

$$\begin{aligned} \int_\Omega |(\partial_t\chi_\varepsilon(x,t))^-|^{\frac{4}{3}}dx &= \int_\Omega (\eta^\varepsilon(x,t))^{\frac{2}{3}}|(\partial_t\chi_\varepsilon(x,t))^-|^{\frac{4}{3}}\frac{1}{(\eta^\varepsilon(x,t))^{\frac{2}{3}}}dx \\ &\le \left\|(\eta^\varepsilon(t))^{\frac{2}{3}}|(\partial_t\chi_\varepsilon(t))^-|^{\frac{4}{3}}\right\|_{L^{3/2}(\Omega)}\left\|\frac{1}{(\eta^\varepsilon(t))^{\frac{2}{3}}}\right\|_{L^3(\Omega)}. \end{aligned} \tag{5.4}$$

Note that the application of Hölder's inequality in the latter passage is justified by the following inequality, due to our assumption (2.3),

$$\frac{1}{(\eta^\varepsilon(x,t))^2} \le k_\eta^{-2}(1 + |\nabla\chi_\varepsilon(x,t)|)^2 \le 2k_\eta^{-2}(1 + |\nabla\chi_\varepsilon(x,t)|^2) \quad \text{for a.e. } (x,t) \in Q.$$

Hence, in view of (5.2), $1/\eta^\varepsilon \in L^\infty(0,T;H)$, and for a.e. $t \in (0,T)$,

$$\begin{aligned} \left\|\frac{1}{(\eta^\varepsilon(t))^{\frac{2}{3}}}\right\|_{L^3(\Omega)} = \left\|\frac{1}{\eta^\varepsilon(t)}\right\|_{L^2(\Omega)}^{\frac{2}{3}} &\le C(1 + \|\nabla\chi_\varepsilon(t)\|_{L^2(\Omega)}^{\frac{2}{3}}) \\ &\le C(1 + \|\chi_\varepsilon\|_{L^\infty(0,T;V)}^{\frac{2}{3}}) \le C. \end{aligned}$$

Thus, it follows from (5.4) that for a.e. $t \in (0,T)$,

$$\|(\partial_t\chi_\varepsilon(t))^-\|_{L^{4/3}(\Omega)} \le C\|\sqrt{\eta^\varepsilon(t)}(\partial_t\chi_\varepsilon(t))^-\|_H,$$

so that (5.3) yields

$$\|(\partial_t\chi_\varepsilon)^-\|_{L^2(0,T;L^{4/3}(\Omega))} \le C \quad \forall\varepsilon > 0. \tag{5.5}$$

Third a priori estimate. Preliminarily, we note that for a.e. $x \in \Omega$ and for all $t \in [0,T]$,

$$\left|\int_0^t \partial_t\chi_\varepsilon(x,s)ds\right| \le |\chi_\varepsilon(x,t)| + |\chi_0^\varepsilon(x)|,$$

so that, by (5.2),

$$\int_\Omega\left|\int_0^T \partial_t\chi_\varepsilon(x,s)ds\right|dx \le |\Omega|^{1/2}\left(\|\chi_\varepsilon\|_{L^\infty(0,T;H)} + \|\chi_0^\varepsilon\|_H\right) \le C.$$

Therefore,

$$\begin{aligned} \|(\partial\chi_\varepsilon)^+\|_{L^1(0,T;L^1(\Omega))} = \int_\Omega\int_0^T(\partial_t\chi_\varepsilon(x,s))^+ &\le \int_\Omega\left|\int_0^T\partial_t\chi_\varepsilon\right| + \int_\Omega\int_0^T(\partial_t\chi_\varepsilon)^- \\ &\le C(1 + \|(\partial_t\chi_\varepsilon)^-\|_{L^2(0,T;L^{4/3}(\Omega))}). \end{aligned}$$

In view of the previous (5.5), we obtain

$$\|\partial_t\chi_\varepsilon\|_{L^1(0,T;L^1(\Omega))} \le C \quad \forall\varepsilon > 0. \tag{5.6}$$

Fourth a priori estimate. By comparison in (2.14), we conclude

$$\|\vartheta_\varepsilon + \chi_\varepsilon\|_{H^1(0,T;V')} \le C \quad \forall \varepsilon > 0. \tag{5.7}$$

Moreover, testing (2.15) by ξ_ε and integrating in time, we find

$$\begin{aligned} &\varepsilon \int_\Omega \widehat{\beta}(\chi_\varepsilon(x,t))dx + \int_0^t \|\xi_\varepsilon(s)\|_H^2 ds \\ &\le \varepsilon \int_\Omega \widehat{\beta}(\chi_0^\varepsilon(x))dx + \int_0^t \left(\eta^\varepsilon(s)(\partial_t\chi_\varepsilon(s))^- - \sigma'(\chi_\varepsilon(s)) + \vartheta_\varepsilon(s), \xi_\varepsilon(s)\right)_H ds. \end{aligned} \tag{5.8}$$

Actually, (5.8) ensues from the chain rule [8, Lemma 3.3, p. 73], and from the formal estimate

$$\int_0^t (A\chi_\varepsilon(s), \xi_\varepsilon(s))_H \, ds \ge 0,$$

which is due to the monotonicity of β and could be made rigorous by approximating β with its Yosida regularization. Exploiting the positivity of $\widehat{\beta}$, (2.20) and the boundedness of $\{\chi_0^\varepsilon\}$ in V, the a priori bound (5.2) (which yields, by the Lipschitz continuity of σ', that $\sigma'(\chi_\varepsilon)$ is bounded in $L^2(0,T;H)$), and, finally, (5.3), we easily deduce that

$$\{\xi_\varepsilon\} \quad \text{is bounded in } L^2(0,T;H). \tag{5.9}$$

Finally, testing (2.15) by $\{A\chi_\varepsilon\}$, using as usual the formal identity

$$(\partial_t\chi_\varepsilon(t), A\chi_\varepsilon(t))_H = \frac{1}{2}\frac{d}{dt}\|\nabla\chi_\varepsilon\|_H^2(t) \quad \text{for a.e. } t \in (0,T),$$

and taking into account all the previous estimates, we conclude that $\{A\chi_\varepsilon\}$ is bounded in $L^2(0,T;H)$, whence, by elliptic regularity results,

$$\|\chi_\varepsilon\|_{L^2(0,T;W)} \le C \quad \forall \varepsilon > 0. \tag{5.10}$$

Compactness. By standard weak-star compactness results, the estimates (5.2) and (5.10) immediately yield that there exist $\chi \in L^\infty(0,T;V) \cap L^2(0,T;W)$ and a subsequence along which (2.23) holds. Combining (5.2), (5.10), and (5.6), by [19, Thm. 5, Cor. 4] we also deduce that $\chi_{\varepsilon_k} \to \chi$ in $L^2(0,T;V)$; the latter convergence and the bound (5.2) immediately yield (2.24) via the Lebesgue theorem. Further, (2.25) is a trivial consequence of (5.2).

As for $\{\vartheta_{\varepsilon_k}\}$, (5.2) yields that there exist $\vartheta \in L^2(0,T;V) \cap L^\infty(0,T;H)$ and a subsequence (which we do not relabel), such that (2.27) holds. On the other hand, thanks to (5.7) the sequence $\{e_{\varepsilon_k}\}$ is weakly compact in $H^1(0,T;V')$. Moreover, by the aforementioned compact results in [19] and estimate (5.2) we have that $\{e_{\varepsilon_k}\}$ is compact in $C^0([0,T];V') \cap L^2(0,T;H)$, hence in $L^p(0,T;H)$ for all $1 \le p < \infty$ via the bound in $L^\infty(0,T;H)$ and the Lebesgue theorem again. In view of (2.23), and (2.27) we easily identify the limit of $\{e_{\varepsilon_k}\}$ to be $\vartheta + \chi$, so that, up to extracting a further subsequence, the convergences (2.29)–(2.30) hold. As it is well known, the regularity $e \in C^0([0,T];V') \cap L^\infty(0,T;H)$ yields $e \in C^0_w([0,T];H)$. Note that (2.30) and (2.24) entail (2.28).

Furthermore, (2.31) is a consequence of (5.9); by the strong convergence of χ_{ε_k} to χ in $L^2(0,T;H)$ and by the strong-weak closedness of the graph of β (more precisely, of the graph of the maximal monotone operator induced by β on $L^2(0,T;H)$), we conclude that $\xi \in \beta(\chi)$ a.e. in Q. Finally, recalling Remark B.5, we infer from (5.6) that the sequence $\partial_t\chi_\varepsilon$ is tight, so that by Theorem B.4 $\partial_t\chi_{\varepsilon_k}$ admits a limiting Young measure $\boldsymbol{\nu} \in \mathcal{Y}(Q;\mathbb{R})$, fulfilling (B.4), which entails (2.21), as well as (B.5).

Proof of (2.26). Now, we fix an arbitrary $j \in L^2(0,T;L^4(\Omega))$ and choose in (B.5) the normal integrand $g : Q\times\mathbb{R} \to (-\infty,+\infty]$ given by $g(x,t,\xi) := j(x,t)(\xi)^-$. Note that the sequence $(x,t) \mapsto g^-(x,t,\partial_t\chi_{\varepsilon_k}(x,t)) = (j(x,t))^-(\partial_t\chi_{\varepsilon_k}(x,t))^-$ is uniformly integrable on Q; in fact, the estimate

$$\begin{aligned}\int_{I\times A} |(j(x,t))^-(\partial_t\chi_{\varepsilon_k}(x,t))^-|dxdt &\le \int_I \|j(t)\|_{L^4(A)}\|(\partial_t\chi_{\varepsilon_k}(t))^-\|_{L^{4/3}(A)}\\ &\le \|(\partial_t\chi_{\varepsilon_k})^-\|_{L^2(0,T;L^{4/3}(\Omega))}\left(\int_I \|j(t)\|^2_{L^4(A)}dt\right)^2 \quad \forall A\subset\Omega,\ I\subset(0,T),\end{aligned}$$

the estimate (5.5) on $(\partial_t\chi_\varepsilon)^-$, and the elementary property

$$\forall\epsilon>0\ \exists\delta>0 \quad \text{s.t. } |I\times A|\le\delta \Rightarrow \|j\|_{L^2(I;L^4(A))}\le\epsilon,$$

easily yield that $\{(j)^-(\partial_t\chi_{\varepsilon_k})^-\}$ complies with the definition of uniform integrability. Hence, by (B.5) we have

$$\begin{aligned}\liminf_{k\uparrow\infty}\int_0^T {}_{L^{4/3}(\Omega)}\langle(\partial_t\chi_{\varepsilon_k}(t))^-, j(t)\rangle_{L^4(\Omega)}\,dt &= \liminf_{k\uparrow\infty}\int_Q j(x,t)\,(\partial_t\chi_{\varepsilon_k}(x,t))^-\,dxdt\\ \ge \int_Q j(x,t)\left(\int_{\mathbb{R}}(\xi)^-d\nu_{(x,t)}(\xi)\right)dxdt &= \int_0^T {}_{L^{4/3}(\Omega)}\langle\ell(t), j(t)\rangle_{L^4(\Omega)}\,dt.\end{aligned}$$

Choosing now in (B.5) the normal integrand $\tilde g(x,t,\xi) := -j(x,t)(\xi)^-$ (it can be checked in the same way that the sequence $(x,t)\mapsto\tilde g^-(x,t,\partial_t\chi_{\varepsilon_k}(x,t)) = (j(x,t))^+(\partial_t\chi_{\varepsilon_k}(x,t))^-$ is uniformly integrable), we easily obtain

$$\begin{aligned}\limsup_{k\uparrow\infty}\int_0^T {}_{L^{4/3}(\Omega)}\langle(\partial_t\chi_{\varepsilon_k}(t))^-, j(t)\rangle_{L^4(\Omega)}\,dt &\le \int_Q j(x,t)\left(\int_{\mathbb{R}}(\xi)^-d\nu_{(x,t)}(\xi)\right)dxdt\\ &= \int_0^T {}_{L^{4/3}(\Omega)}\langle\ell(t), j(t)\rangle_{L^4(\Omega)}\,dt.\end{aligned}$$

Hence, we conclude (2.26). Combining this with (2.24), we observe that (2.34) is satisfied.

In the end, note that

$$\eta(\vartheta_{\varepsilon_k},\nabla\chi_{\varepsilon_k})(\partial_t\chi_{\varepsilon_k})^- \rightharpoonup \eta(\vartheta,\nabla\chi)\ell \quad \text{in } L^2(0,T;V'),\text{ as } k\uparrow\infty. \tag{5.11}$$

In fact, up to extracting further subsequences, we deduce from (2.24) and (2.28) that $\vartheta_{\varepsilon_k}\to\vartheta$ and $\nabla\chi_{\varepsilon_k}\to\nabla\chi$ a.e. on Ω. Arguing as in the previous section, we conclude by the Lebesgue theorem that

$$\eta(\vartheta_{\varepsilon_k},\nabla\chi_{\varepsilon_k})\to\eta(\vartheta,\nabla\chi) \quad \text{in } L^p(0,T;L^q(\Omega)) \text{ for all } 1\le p,q<\infty,$$

and it is then easy to check (5.11), taking into account (2.26).

Passage to the limit. The convergences (2.23)–(2.31) so far obtained, as well as (5.11) and (2.19), enable us to pass to the limit as $\varepsilon_k\downarrow 0$ in (2.14), (2.15) and in the initial conditions (2.16) (recalling (2.17)). Hence, the pair (ϑ,χ) fulfills (2.14), (2.33) and (2.32).

By (5.6), we also conclude that, up to a subsequence, $\partial_t\chi_{\varepsilon_k}$ weakly star converges to a Radon measure $\mu\in M(Q)$, which we can identify with the *distributional derivative*

$\partial_t \chi$ of χ. In view of Remark B.5, we may compare μ and the limit Young measure $\boldsymbol{\nu}$. Indeed, introducing the measure ρ (cf. (2.35)), we deduce (2.36), which states that the measure $\mu - \rho$ is positive. □

Appendix A. Mosco and G-convergence. We refer to, e.g., the monograph [4] for an exhaustive exposition of the notions which we are briefly recalled below. Throughout this subsection, $\mathscr{H}$ will denote a Hilbert space, with scalar product $\langle \cdot, \cdot \rangle$.

Definition A.1 (Mosco convergence). Let $\psi_n, \psi : \mathscr{H} \to \mathbb{R} \cup \{+\infty\}$ be proper, convex, and l.s.c. functionals: we say that $\{\psi_n\}$ converges to ψ *in the sense of Mosco* if

- $\forall z \in \mathscr{H}$ there exists a sequence $z_n \to z$ such that $\psi_n(z_n) \to \psi(z)$ as $n \uparrow \infty$;
- $\forall z \in \mathscr{H}$ and $\forall z_n \rightharpoonup z$ as $n \uparrow \infty$, $\psi(z) \leq \liminf_{n\uparrow\infty} \psi_n(z_n)$.

As a straightforward consequence of [4, Prop. 3.20, p. 298], we have that for every proper (convex and l.s.c) functional $\psi : \mathscr{H} \to \mathbb{R} \cup \{+\infty\}$, the sequence of the Moreau-Yosida approximates $\{\psi_\lambda\}_\lambda$ of ψ,

$$\psi_\lambda \text{ Mosco-converges to } \psi \text{ as } \lambda \downarrow 0. \tag{A.1}$$

Definition A.2 (G-convergence). We say that a sequence $\mathscr{A}^n : \mathscr{H} \to 2^{\mathscr{H}}$ of maximal monotone operators converges to a maximal monotone operator $\mathscr{A}$ on $\mathscr{H}$ *in the sense of G-convergence* (or *in the sense of graphs*), if $\forall [x, y] \in \mathscr{A}$ there exists a sequence $[x_n, y_n] \in \mathscr{A}^n$ such that $[x_n, y_n] \to [x, y]$ strongly in $\mathscr{H} \times \mathscr{H}$.

A crucial property of G-convergence (which can be retrieved in the proof of [4, Prop. 3.59, p. 361]) is that, when $\mathscr{A}^n$ G-converges to $\mathscr{A}$, then

$$\begin{cases} [x_n, y_n] \in \mathscr{A}^n, \quad x_n \rightharpoonup x, \quad y_n \rightharpoonup y \quad \text{in } \mathscr{H}, \\ \liminf_{n\uparrow\infty} \langle x_n, y_n \rangle \leq \langle x, y \rangle \end{cases} \Longrightarrow [x, y] \in \mathscr{A}. \tag{A.2}$$

Appendix B. Compactness tools of Young measures theory. We briefly recall some basic notions and results of Young measures theory, referring, e.g., to [20, 6] for a self-contained introduction to this topic.

Notation. In the sequel, B will be a separable Banach space and Q the product space $\Omega \times (0, T)$; $\mathcal{L}$ and $\mathcal{B}$ will denote the σ-algebras of the Lebesgue measurable subsets of Q and of the Borel subsets of B, respectively, and $\mathcal{L} \otimes \mathcal{B}$ the usual product σ-algebra in the space $Q \times B$. Further, the set of all Borel probability measures on B is denoted by $\mathcal{P}(B)$, while $C^b(B)$ will be the Banach space of the continuous and bounded real functions defined on B and $\mathcal{M}(Q; B)$ the set of measurable functions from Q to B.

We recall that a function $h : Q \times B \to [0, +\infty]$ is a *positive normal integrand* if

$$h : Q \times B \to [0, +\infty] \quad \text{is } \mathcal{L} \otimes \mathcal{B}\text{-measurable}, \tag{B.1a}$$

$$\text{the maps } v \mapsto h_{(x,t)}(v) := h(x, t, v) \text{ are l.s.c. for a.e. } (x, t) \in Q. \tag{B.1b}$$

A positive normal integrand h is also *coercive* if the sublevels

$$\big\{ v \in B : h_{(x,t)}(v) \leq c \big\} \text{ are compact for any } c \geq 0 \text{ and for a.e. } (x, t) \in Q. \tag{B.1c}$$

Definition B.1 (Young measures). A *Young measure* is a family $\boldsymbol{\nu} := \{\nu_{(x,t)}\}_{(x,t)\in Q}$ of probability measures in $\mathcal{P}(B)$, such that

$$(x,t) \in Q \mapsto \int_B f(\xi)\, d\nu_{(x,t)}(\xi) \quad \text{is} \quad \mathcal{L}\text{-measurable} \quad \forall f \in C^b(B).$$

We denote by $\mathcal{Y}(Q; B)$ the set of all Young measures.

We recall a version of Fubini's Theorem, adapted to families of Young measures [12, p. 20-II].

Theorem B.2. Let $\boldsymbol{\nu} = \{\nu_{(x,t)}\}_{(x,t)\in Q}$ be a Young measure in B; there exists one and only one measure ν on $\mathcal{L} \otimes \mathcal{B}$ such that

$$\nu(A \times C) = \int_A \nu_{(x,t)}(C)\, dxdt \quad \forall\, A \in \mathcal{L},\ C \in \mathcal{B};$$

in particular, $\nu(A \times B) = |A| \quad \forall\, A \in \mathcal{L}$. Moreover, for every $\mathcal{L} \otimes \mathcal{B}$-measurable function $h : Q \times B \to [0, +\infty]$, the function

$$(x,t) \mapsto \int_B h(x,t,\xi) d\nu_{(x,t)}(\xi) \quad \text{is } \mathcal{L}\text{-measurable},$$

and the following extension of Fubini's formula holds

$$\int_{Q\times B} h(x,t,\xi)\, d\nu(x,t,\xi) = \int_Q \Big(\int_B h(x,t,\xi) d\nu_{(x,t)}(\xi) \Big)\, dxdt. \tag{B.2}$$

Definition B.3 (Tightness). We say that a family $\mathcal{U} \subset \mathcal{M}(Q; B)$ is *tight* w.r.t. a normal coercive integrand h satisfying (B.1a, b, c) if

$$S := \sup_{u\in\mathcal{U}} \int_Q h(x,t,u(x,t)) dt < +\infty. \tag{B.3}$$

We say that $\mathcal{U}$ is tight in B if there exists a normal coercive integrand h for which (B.3) holds.

Finally, we recall that a family $\mathcal{U} \subset L^1(0,T; B)$ is *uniformly integrable* if

$$\forall\, \varepsilon > 0 \quad \exists\, \delta > 0 : \quad \forall\, J \subset (0,T) \quad |J| < \delta \ \Rightarrow\ \sup_{u\in\mathcal{U}} \int_J \|u(t)\|_B\, dt \le \varepsilon.$$

The following crucial compactness result was first proved in [5].

Theorem B.4 (Balder). Let $u^n \in \mathcal{M}(Q; B)$ be tight w.r.t. a normal coercive integrand. Then, there exists a subsequence u^{n_k} and a Young measure $\boldsymbol{\nu} = \{\nu_{(x,t)}\}_{(x,t)\in Q}$ in $\mathcal{Y}(Q; B)$, which we call a *limit Young measure for* u^n, such that for a.e. $(x,t) \in Q$,

$$\operatorname{supp}(\nu_{(x,t)}) \subset \cap_{p=1}^{\infty} \overline{\{u^{n_k}(x,t) : k \ge p\}}, \tag{B.4}$$

(i.e., the measure $\nu_{(x,t)}$ is concentrated on the set of the limit points of $\{u^{n_k}(x,t)\}$), and

$$\begin{gathered}\liminf_{k\to\infty} \int_Q g(x,t,u^{n_k}(x,t))\, dxdt \ge \int_Q \left(\int_E g(x,t,\xi) d\nu_{(x,t)}(\xi) \right) dxdt \\ \text{for every normal integrand } g : Q \times B \to (-\infty, +\infty] \text{ s.t.} \\ \text{the sequence } (x,t) \mapsto g^-(x,t,u^{n_k}(x,t)) \text{ is uniformly integrable.}\end{gathered} \tag{B.5}$$

Remark B.5 (Comparison between limits in the sense of measures). For later convenience, let us focus on the case $B := \mathbb{R}$, and let $\{u^n\} \subset L^1(Q)$ be a bounded sequence, i.e.,

$$\sup_{n\in\mathbb{N}} \int_Q |u^n(x,t)|dxdt < +\infty. \tag{B.6}$$

It follows from well-known weak compactness results in functional analysis that $\{u^n\}$ admits a subsequence $\{u^{n_k}\}$ weakly-star converging to a measure μ in the space $M(Q)$ of the Radon measures on Q, i.e.,

$$\lim_{k\uparrow\infty} \int_Q u^{n_k}(x,t)f(x,t)dxdt = \langle \mu, f\rangle \quad \forall f \in C_0(Q), \tag{B.7}$$

$C_0(Q)$ denoting the space of the continuous functions on Q with compact support.

On the other hand, (B.6) is a *tightness estimate*, as the functional $h(x,t,\xi) := |\xi|$ is trivially a normal coercive integrand on $Q \times \mathbb{R}$. Therefore, by Theorem B.4, there exists a limit Young measure $\boldsymbol{\nu}$ such that, up to a subsequence, (B.5) holds.

In particular, if the sequence $\{(u^n)^-\}$ is uniformly integrable, it follows from (B.5) that

$$\liminf_{k\uparrow\infty} \int_Q f(x,t)u^{n_k}(x,t)dxdt \geq \int_Q f(x,t)\left(\int_{\mathbb{R}} \xi d\nu_{(x,t)}(\xi)\right)dxdt$$

for all *positive* $f \in C_0(Q)$ (it suffices to apply (B.5) to the integrand $g(x,t,\xi) := f(x,t)\xi$, and note that, since $f \geq 0$, $g^-(x,t,u^n(x,t)) = f(x,t)(u^n)^-(x,t)$ for a.e. $(x,t) \in Q$). Let us now denote by ϱ the Radon measure on Q defined by

$$\langle \varrho, f\rangle := \int_Q f(x,t)\left(\int_{\mathbb{R}} \xi d\nu_{(x,t)}(\xi)\right)dxdt. \tag{B.8}$$

Hence, in view of (B.7), we conclude

$$\langle \mu, f\rangle \geq \langle \varrho, f\rangle \quad \forall f \in C_0(Q),\ f \geq 0. \tag{B.9}$$

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