# GLOBAL ATTRACTOR FOR THE WEAK SOLUTIONS OF A CLASS OF VISCOUS CAHN-HILLIARD EQUATIONS

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We address the long-time behaviour of a class of viscous Cahn-Hilliard equations, modelling phase separation in mixtures and alloys. Specifically, we prove the existence of (a suitable notion of) the global attractor for the *weak solutions* of the so-called *generalized* viscous Cahn-Hilliard equation.

## 1. Introduction

This paper is concerned with the analysis of the long-time behavior of the (weak) solutions of the following fourth-order equation

$$\partial_t \chi - \Delta(\alpha(\partial_t \chi - \Delta\chi + \chi^3 - \chi)) = 0 \quad \text{in } \Omega \times (0, +\infty). \tag{1}$$

Here, $\Omega$ a bounded, connected domain in $\mathbb{R}^N$, $N = 1, 2, 3$, with smooth boundary $\Gamma$; $\alpha : D(\alpha) \subset \mathbb{R} \to \mathbb{R}$ is a (strictly) increasing, differentiable function, while the term $\chi^3 - \chi$ is the derivative of the double-well potential

$$\mathcal{W}(x) = \frac{(x^2 - 1)^2}{4}, \quad x \in \mathbb{R}. \tag{2}$$

In fact, (1) models the evolution of a phase separation process, to which a two-phase material (for instance, a binary alloy or a mixture), occupying the domain $\Omega$, is subject. In this connection, the variable $\chi$, usually referred to as *order parameter*, stands for the local concentration of one of the two components.

Equation (1) is indeed a *generalized* viscous Cahn-Hilliard equation: in fact, the viscous Cahn-Hilliard equation

$$\partial_t \chi - \kappa\Delta(\partial_t \chi - \Delta\chi + \chi^3 - \chi) = 0 \quad \text{in } \Omega \times (0, +\infty) \tag{3}$$

can be retrieved from (1) by choosing $\alpha(r) := \kappa r$ for every $r \in \mathbb{R}$, $\kappa > 0$ being the *mobility coefficient*.

On the other hand, (3) is itself a viscous regularization of the well-known Cahn-Hilliard equation

$$\partial_t \chi - \kappa \Delta(-\Delta \chi + \chi^3 - \chi) = 0 \quad \text{in } \Omega \times (0, +\infty), \tag{4}$$

originally introduced in the paper [6], dating back to 1961, for modelling phase separation phenomena driven by a quick quenching. Almost thirty years later, (3) was proposed in [16] to account for viscosity effects in the phase separation of polymeric systems. We refer to the survey [17] for a detailed overview of several analytical results on equations on (4) and (3), especially concerning well-posedness issues, and the dynamics of pattern formation. The asymptotic behavior as $t \to +\infty$ of the solutions to (suitable boundary value problems for) (4) and (3) has also been extensively tackled. In particular, we mention the seminal papers [9], [15] (see also Chap. III in [25]), for (4), and [8] for (3).

The generalized viscous Cahn-Hilliard equation (1) has instead been proposed in the much more recent paper [11], in the framework of a new approach to the modeling of phase separation phenomena. The key idea in [11] is that the work of the internal microforces, accompanying the changes of $\chi$, should be taken into account. On these grounds, a new, unified derivation (also expounded in [12], [13]) of (4) and (3) is developed, leading to the *generalized Cahn-Hilliard equation*

$$\partial_t \chi - \operatorname{div}(\mathbf{M}(Z) \nabla (\delta \partial_t \chi - \Delta \chi + \mathcal{W}'(\chi))) = 0 \quad \text{in } \Omega \times (0, +\infty). \tag{5}$$

Here, $\delta \geq 0$ is a positive constant ($\delta > 0$ in the viscous case), and $Z$ denotes the set of the independent constitutive variables the mobility tensor $\mathbf{M}$ (in general, a positive definite $N \times N$ matrix), is allowed to depend on. We refer to [12], [13], to the survey [14], to [19], and the references therein, for a detailed account of the results on well-posedness and long-time behavior of several kinds of Cahn-Hilliard equations derived from (5).

As far as (1) is concerned, its well-posedness has been indeed first tackled in [19], where it has also been shown that (1) can be obtained as a particular case of (5). Indeed, it is sufficient to choose a mobility tensor $\mathbf{M}$ only depending on the *modified* chemical potential

$$w = \partial_t \chi - \Delta \chi + \chi^3 - \chi,$$

through the formula $\mathbf{M} = \mathbf{M}(w) := \alpha'(w)\mathbf{I}$. Note that in [19] (1) has been formulated in the variables $\chi$ and $u := \alpha(w)$, which splits (1) into the system

$$\begin{aligned} \partial_t \chi - \Delta u = 0 \quad \text{in } \Omega \times (0, +\infty), \\ \partial_t \chi - \Delta \chi + \chi^3 - \chi = \rho(u) \quad \text{in } \Omega \times (0, +\infty), \end{aligned} \tag{6}$$

where $\rho$ is the inverse of $\alpha$. Further, two different choices for the nonlinearity $\alpha$ and, accordingly, two different choices for the boundary conditions on $\chi$ and $u$, have been considered therein. Indeed, the well-posedness of the initial-boundary value problem for (6) (supplemented with a source term $f$ in the first equation), on a *finite time interval* $(0, T)$, has been first tackled in the case of

$$\text{a \textit{bi-Lipschitz}, strictly increasing function } \alpha_1 : \mathbb{R} \to \mathbb{R}, \tag{7}$$

coupled with *Neumann* boundary conditions

$$\partial_n \chi = 0 \quad \text{and} \quad \partial_n u = 0 \quad \text{in } \Gamma \times (0, T), \tag{8}$$

on $\chi$ and $u$ (non-homogeneous Neumann boundary conditions on $u$ have also been considered). Secondly, the well-posedness of (6) on $(0, T)$ has been addressed in the case of a

$$\begin{aligned} &\text{a strictly increasing function } \alpha_2 : (a, +\infty) \to \mathbb{R}, \lim_{r \downarrow a} \alpha_2(r) = -\infty, \\ &\quad \text{with inverse } \rho_2 := \alpha_2^{-1} \text{ Lipschitz (but \textit{not bi-Lipschitz})}. \end{aligned} \tag{9}$$

In the latter case, (6) has been supplemented with a homogeneous Neumann boundary condition on $\chi$, and with a third type (or Robin) condition on $u$, i.e.

$$\partial_n \chi = 0 \quad \text{and} \quad -\partial_n u = \omega u \quad \text{in } \Gamma \times (0, T), \tag{10}$$

$\omega$ being a positive constant.

We refer to [19] for further modeling details, also motivating the choices (7) and (9) of the nonlinearities. Instead, we want to stress the main difficulties connected with the analysis of (6), both in the case of the choices (7)-(8), and of (9)-(10). Roughly speaking, due to the presence of the nonlinearity $\rho$ in the second equation of (6), the *a priori* bounds on (suitable norms of) $u$ (needed to pass to the limit in, possibly, an approximation procedure), can only be obtained through careful estimates. In the case of (7)-(8), these computations substantially rely on the fact that $\rho_1$ is bi-Lipschitz. On the other hand, in the case of (9)-(10) - when only the Lipschitz continuity of $\rho$ is assumed -, the third type condition (10) on $u$ plays a key role. Thus, well-posedness and, in the case of (9)-(10), also regularity results have been obtained in [19] for both boundary-value problems, supplemented with an initial datum $\chi_0 \in H^1(\Omega)$. In view of such results, the related solution operators form a *strongly continuous* semigroup on the phase space $H^1(\Omega)$. The existence of the universal attractor in this regularity framework shall be investigated in the paper [20].

Here, paralleling the analysis of [15] (see also [25]), we shall instead focus on the existence and the long-time behaviour of a class of *weak solutions* to (6). Such solutions in fact emanate from initial data for $\chi$ in the *bigger* phase space $L^4(\Omega)$ - note that such a choice is linked to the natural domain of the nonlinearly $\mathcal{W}$ (2), as in the approach of [18].

In this weaker regularity setting, the analytical drawbacks due to the nonlinearity $\rho$ become more patent, and are less easy to overcome. Indeed (see Theorem 8 later on), we have been able to prove the existence of solutions to the initial-boundary value problem for (6), with an initial datum $\chi_0 \in L^4(\Omega)$ only when with the *same* kind of boundary condition on $\chi$ and $u$ is considered. We shall denote by $J$ the operator realizing $-\Delta$ in the first and in the second of (6) with such a boundary condition. Hence, the key assumption is that $J$ be coercive w.r.t. the $H^1(\Omega)$-norm. For example, we may tackle within this framework homogeneous Dirichlet conditions on $\chi$ and $u$

$$\chi = u = 0 \quad \text{in } \Gamma \times (0, +\infty) \tag{11}$$

or third type boundary conditions on $\chi$ and $u$, namely

$$\partial_n \chi + \omega \chi = 0 \quad \text{and} \quad \partial_n u + \omega u = 0 \quad \text{in } \Gamma \times (0, +\infty), \tag{12}$$

$\omega$ being a positive constant.

Note that homogeneous *Neumann* boundary conditions on $u$ and $\chi$ are more usual in the framework of the Cahn-Hilliard equations. Nonetheless (cf. [13]), Dirichlet conditions on $u$ might for instance be considered in the case in which (6) models the propagation of a solidification front in a medium at rest with respect to the front; furthermore, the homogeneous Dirichlet conditions (11) have been already considered in [8] and [26].

Still, even in the setting of these easier-to-handle conditions, no uniqueness/continuous dependence results are available: (6) does not generate a semigroup, and the standard theory of [25] cannot be directly applied. Thus, we have exploited the new theory of *generalized semiflows*, recently proposed by J. M. Ball in [1], [2] for the study of the long-time behavior of solutions of differential problems lacking uniqueness. Let us stress that this theory is also related to the approach of *trajectory attractors* developed in [7] (see also [21, 22]). For instance, Ball's approach has been successfully applied to the study of the long-time behavior of the solutions of the Navier-Stokes equation [1], and of the semilinear damped wave equation [3].

In fact, we have shown (cf. Theorem 10) that the set of the *weak solutions* has the structure of a generalized semiflow on the phase space $L^4(\Omega)$, according to the definition given in [1] (see Section 2). Hence, we have proved in Theorem 11 that the generalized semiflow of the *weak solutions* possesses a unique *global attractor*.

**Plan of the paper.** In Section 2, for the reader's convenience we briefly recall the main definitions and results on *generalized semiflows* and their long-time behavior developed in the papers [1], [2], of which we closely follow the outline. In Section 3, we introduce the definition of weak solution to (1) by means of a suitable variational formulation, which in fact subsumes several kinds of "coercive" boundary conditions (like (11) and (12)), see Remarks 4 and 5. Hence, we state our main theorems, whose proof is developed in Section 4.

## 2. Preliminaries: generalized semiflows

**Definition of generalized semiflow.** Let $(\mathcal{X}, d_{\mathcal{X}})$ be a (not necessarily complete) metric space; we recall that the *Hausdorff semidistance* $e(A,B)$ of two subsets $A,\ B \subset \mathcal{X}$ is given by $e(A,B) := \sup_{a\in A} \inf_{b\in B} d_{\mathcal{X}}(a,b)$, while the *Hausdorff distance* $\operatorname{dist}(A,B)$ of $A$ and $B$ is defined by $\operatorname{dist}(A,B) := \max\{e(A,B), e(B,A)\}$.

**Definition 1:** A *generalized semiflow* $\mathcal{S}$ on $\mathcal{X}$ is a family of maps $u : [0,+\infty) \to \mathcal{X}$ (referred to as "solutions"), satisfying:

**(H1)** *(Existence)* for any $v \in \mathcal{X}$ there exists at least one $u \in \mathcal{S}$ with $u(0) = v$;

**(H2)** *(Translates of solutions are solutions)* for any $u \in \mathcal{S}$ and $\tau \geq 0$, the map $u^\tau(t) := u(t+\tau)$, $t \in [0,+\infty)$, is in $\mathcal{S}$;

**(H3)** *(Concatenation)* for any $u$, $w \in \mathcal{S}$ and $t \geq 0$ with $w(0) = u(t)$, then $z \in \mathcal{S}$, $z$ being the map defined by

$$z(\tau) := \begin{cases} u(\tau) & \text{if } 0 \leq \tau \leq t, \\ w(\tau - t) & \text{if } t < \tau; \end{cases}$$

**(H4)** *(Upper-semicontinuity w.r.t. initial data)* if $\{u_n\} \subset \mathcal{S}$ and $u_n(0) \to v$, then there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$, and $u \in \mathcal{S}$, such that $u(0) = v$ and $u_{n_k}(t) \to u(t)$ for all $t \geq 0$.

**Continuity properties.** A generalized semiflow may enjoy the following properties:

**(C1)** Each $u \in \mathcal{S}$ is continuous from $(0, +\infty) \to \mathcal{X}$;
**(C2)** for any $\{u_n\} \subset \mathcal{S}$ with $u_n(0) \to v$, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$, and $u \in \mathcal{S}$, such that $u(0) = v$ and $u_{n_k} \to u$ uniformly on the compact subsets of $(0, +\infty)$;
**(C3)** Each $u \in \mathcal{S}$ is continuous from $[0, +\infty) \to \mathcal{X}$;
**(C4)** for any $\{u_n\} \subset \mathcal{S}$ with $u_n(0) \to v$, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$, and $u \in \mathcal{S}$, such that $u(0) = v$ and $u_{n_k} \to u$ uniformly on the compact subsets of $[0, +\infty)$.

Of course, (C3) $\Rightarrow$ (C1) and (C4) $\Rightarrow$ (C2).

In addition, the notions of *(positive) orbit* and *$\omega$-limit* (both of a solution $u \in \mathcal{S}$ and of a subset $E \subset \mathcal{X}$), which are classical within the theory of universal attractors for dynamical systems (cf. [25]), can be extended to this multivalued setting. In the same way, the *attraction* and *invariance* properties can be suitably introduced. We refer to [1], Sec. 3, for all the precise definitions, which eventually lead to the definition of *global attractor*. Quoting [1], we say that a set $\mathcal{A} \subset \mathcal{X}$ is a *global attractor* for $\mathcal{S}$ if $\mathcal{A}$ is compact, invariant, and attracts all the bounded sets of $\mathcal{X}$.

**Compactness and dissipativity.** By definition, a generalized semiflow $\mathcal{S}$ is *eventually bounded* if for every bounded $B \subset \mathcal{X}$ there exists $\tau \geq 0$ such that $\gamma^\tau(B)$ is bounded; *point dissipative* if there exists a bounded set $B_0 \subset \mathcal{X}$ such that for any $u \in \mathcal{S}$ there exists $\tau \geq 0$ such that $u(t) \in B_0$ for all $t \geq \tau$; *asymptotically compact* if for any sequence $\{u_n\} \subset \mathcal{S}$ such that $\{u_n(0)\}$ is bounded and for any sequence $t_n \uparrow \infty$, the sequence $\{u_n(t_n)\}$ admits a convergent subsequence; *compact* if for any sequence $\{u_n\} \subset \mathcal{S}$ with $\{u_n(0)\}$ bounded there exists a subsequence $\{u_{n_k}\}$ such that $\{u_{n_k}(t)\}$ is convergent for any $t > 0$.

**Remark 2:** It has also been shown (cf. Prop. 3.2 in [2]), that if $\mathcal{S}$ is eventually bounded and compact, then it is also asymptotically compact.

Finally, we say that a global attractor $\mathcal{A}$ for $\mathcal{S}$ is *Lyapunov stable* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $E \subset \mathcal{S}$ with $e(E, \mathcal{A}) < \delta$, then $e(T(t)E, \mathcal{A}) < \varepsilon$ for all $t \geq 0$.

**Existence of the attractor.** In the end, we recall a criterion (cf. Thms. 3.3 and 6.1 in [1]) for the existence of a global attractor $\mathcal{A}$ for $\mathcal{S}$.

**Theorem 3:** A generalized semiflow $\mathcal{S}$ has a global attractor if and only if it is point dissipative and asymptotically compact; in that case, the attractor $\mathcal{A}$ is unique, it is the maximal compact invariant subset of $\mathcal{X}$, and it can be characterized as

$$\mathcal{A} = \cup\{\omega(B) \ : \ B \subset \mathcal{X}, \text{ bounded}\} = \omega(\mathcal{X}).$$

Moreover, if $\mathcal{S}$ complies with (C1) and (C4), $\mathcal{A}$ is Lyapunov stable.

## 3. Main results

**Notation.** We denote by $H$ the space $L^2(\Omega)$, ($(\cdot,\cdot)_H$ will be the scalar product and $\|\cdot\|_H$ the norm of $H$), while $V$ will denote a Hilbert space $V \subset H^1(\Omega)$, endowed with the norm $\|\cdot\|_V$ of $H^1(\Omega)$; $\|\cdot\|_{V'}$ will denote the norm on $V'$, and $\langle\cdot,\cdot\rangle$ both the duality pairing between $V'$ and $V$, and between $(H^1(\Omega))'$ and $H^1(\Omega)$. Hence,

$$\begin{gathered} (V, H, V') \text{ is a Hilbert triplet, and} \\ V \subset H \equiv H' \subset V' \quad \text{with dense and compact embeddings.} \end{gathered} \tag{13}$$

We also consider a continuous and symmetric bilinear form $j$ on $H^1(\Omega) \times H^1(\Omega)$, and the associated operator $J : H^1(\Omega) \to (H^1(\Omega))'$. We will denote by the same symbol $J$ also its restriction to $V$. We will suppose that

$$\begin{gathered} J \text{ is bounded, linear, symmetric, and coercive on } V, \text{ i.e.,} \\ \exists \gamma > 0 \ : \ \langle Ju, u\rangle \geq \gamma \|u\|_V^2 \ \forall u \in V. \end{gathered} \tag{14}$$

In the sequel, we will also assume that, for any differentiable non decreasing function $h : \mathbb{R} \to \mathbb{R}$, with $h(0) = 0$,

$$\langle J(h(v)), v\rangle \geq 0 \quad \forall v \in D_V(h) := \left\{v \in V \ : \ h(v) \in H^1(\Omega)\right\}. \tag{15}$$

**Remark 4:** For example, we may choose as $V$ the space

$$H^1_{\Gamma_0}(\Omega) := \left\{v \in H^1(\Omega) \ : \ v|_{\Gamma_0} = 0\right\} \tag{16}$$

(where $\Gamma_0$ is a measurable subset of $\Gamma$, with positive measure, and $v|_{\Gamma_0}$ is the trace of $v$ on $\Gamma_0$). Hence, an admissible choice for $J$ is

$$\langle Ju, v\rangle := \int_\Omega \nabla u \nabla v \, dx \quad \forall u, v \in H^1_{\Gamma_0}(\Omega). \tag{17}$$

Another possibility is to choose

$$V := H^1(\Omega) \quad \text{and} \quad \langle Ju, v\rangle := \int_\Omega \nabla u \cdot \nabla v + \omega\langle u, v\rangle_\Gamma \quad \forall u, v \in V, \tag{18}$$

with $\omega > 0$, where $\langle \cdot, \cdot \rangle_\Gamma$ is the duality pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$. In fact, in both cases the coercivity (14) follows from Poincaré's inequality, and (15) trivially holds.

It follows from (14) $J$ has a bounded inverse $J^{-1} : V' \to V$; henceforth, will consider on the space $V$ ($V'$, respectively), the scalar product $(\!(v_1, v_2)\!) := j(v_1, v_2) = \langle Jv_1, v_2 \rangle =$ for every $v_1$, $v_2 \in V$ (the scalar product $(\!(w_1, w_2)\!)_* := j(J^{-1}(w_1), J^{-1}(w_2)) = \langle w_1, J^{-1}(w_2) \rangle$ for every $w_1$, $w_2 \in V'$, resp.). Accordingly, we will endow $V$ and $V'$ with the norms

$$\|v\|_V^2 := \langle Jv, v \rangle \quad \forall v \in V, \quad \|w\|_{V'}^2 := \langle w, J^{-1}(w) \rangle \quad \forall w \in V', \tag{19}$$

which are equivalent to the standard norms of $H^1(\Omega)$ and of $(H^1(\Omega))'$. In particular, $J : V \to V'$ is an isometry. We will also make use of the relation

$$\langle Jw, J^{-1}(v) \rangle = (w, v)_H \quad \forall w \in V, v \in H. \tag{20}$$

Finally, given a Banach space $Y$, $C_w^0([0,T];Y)$ will denote the space of the weakly continuous $Y$-valued functions on $[0,T]$.

**Assumptions on the data.** We suppose that $\alpha : (a, +\infty) \to \mathbb{R}$, $a \in \mathbb{R}$, is a differentiable and strictly increasing function, with

$$m := \inf_{r > a} \alpha'(r) > 0, \quad \text{and} \quad \lim_{r \downarrow a^+} \alpha(r) = -\infty; \tag{21}$$

Hence, its inverse function $\rho$ is defined on the whole real line (up to a translation, we can suppose $\rho(0) = 0$), and

$$\begin{gathered}\rho : \mathbb{R} \to \mathbb{R} \text{ is strictly increasing, differentiable and Lipschitz,} \\ \text{with Lipschitz constant } \frac{1}{m}.\end{gathered} \tag{22}$$

**Statement of the problem.** First of all, we give a variational formulation of the initial-boundary value problem for (1) on the finite time interval $(0,T)$. We focus on the autonomous case, since we are interested in the analysis of the long-time behavior of the solutions. In this connection, towards the construction of a *generalized semiflow*, we shall also introduce the notion of *weak solution* to (6) on the half-line $(0, +\infty)$.

Problem 1: *Given* $\chi_0 \in H$, *find* $\chi \in H^1(0,T;V') \cap L^2(0,T;V) \subset$

$C^0([0,T];H)$ *and* $u \in L^2(0,T;V)$ *fulfilling*

$$\partial_t \chi + Ju = 0 \quad \text{in } V', \quad \text{for a.e. } t \in (0,T), \tag{23}$$

$$\partial_t \chi + J\chi + \chi^3 - \chi = \rho(u) \quad \text{in } V', \quad \text{for a.e. } t \in (0,T), \tag{24}$$

$$\chi(x,0) = \chi_0(x) \quad \text{for a.e. } x \in \Omega. \tag{25}$$

**Remark 5:** Let us stress that the variational formulation of Problem 1 entails the same boundary conditions on $\chi$ and $u$, depending on the choice of the space $V$ and of the operator $J$. For example, the choices (16) for $V$ and $J := A|_V$ yield the homogeneous Dirichlet conditions (11); the choice (18) yields the third type conditions (12).

Now, for any pair of functions $\chi : [0,+\infty) \to L^4(\Omega)$ and $u : [0,+\infty) \to V$, let us set for a.e. $t \in (0,+\infty)$

$$\begin{aligned} \mathcal{V}(\chi,u)(t) := {} & \frac{1}{4}\|\chi(t)\|^4_{L^4(\Omega)} \\ & + \int_0^t \Big( \|\chi^3(\tau)\|^2_H - \|\chi(\tau)\|^4_{L^4(\Omega)} - \big(\rho(u(\tau)), \chi^3(\tau)\big)_H \Big) \, d\tau. \end{aligned} \tag{26}$$

We are now in the position to give the following

**Definition 6:** A function $\chi : [0,+\infty) \to L^4(\Omega)$ is called a *weak solution* to (6) if there exists $u : [0,+\infty) \to V$ such that for any $T > 0$

$$\begin{cases} \chi \in H^1(0,T;V') \cap L^2(0,T;V) \cap C^0_w([0,T];L^4(\Omega)) \cap L^6(0,T;L^6(\Omega)) \\ u \in L^2(0,T;V), \end{cases} \tag{27}$$

the pair $(\chi, u)$ fulfils

$$\partial_t \chi + Ju = 0 \quad \text{in } V', \quad \text{for a.e. } t \in (0,+\infty), \tag{28}$$

$$\partial_t \chi + J\chi + \chi^3 - \chi = \rho(u) \quad \text{in } V', \quad \text{for a.e. } t \in (0,+\infty), \tag{29}$$

and there exists a negligible set $\mathcal{N} \subset (0,T]$ such that $(\chi,u)$ complies with

$$\mathcal{V}(\chi,u)(t) \le \mathcal{V}(\chi,u)(s) \quad \forall t \in [0,T], \quad \forall\, s \in [0,t] \setminus \mathcal{N}. \tag{30}$$

We denote by $\mathcal{S}_w$ the set of all weak solutions.

Note that for any weak solution $(\chi,u)$, the map $t \mapsto \mathcal{V}(u,\chi)(t)$ is in $L^\infty_{loc}(0,+\infty)$ thanks to the regularity (27). In the sequel (see e.g. the statement of Theorem 8), with a slight abuse of notation we shall also call *weak solution* any solution to Problem 1, on the finite-time interval $[0,T]$, having the regularity (27) and complying with the inequality (30).

**Remark 7:** Definition 6 highlights the auxiliary role of the variable $u$. For simplicity, in the sequel we will sometimes happen to call the *pair* $(\chi, u)$ weak/strong solution. Nonetheless, the most relevant solution component, also in view of the long-time behavior analysis, is in fact $\chi$.

**Existence of weak solutions.**

**Theorem 8:** Assume (13), (14), (15), (21), and

$$m > \frac{1}{3}. \tag{31}$$

Then, for any

$$\chi_0 \in L^4(\Omega) \tag{32}$$

Problem 1 admits a *weak solution* $\chi$ (in the sense of Definition 6), enjoying the *further regularity*

$$\chi \in C^0([0,T]; L^4(\Omega)). \tag{33}$$

**Remark 9:** Trivial changes in the proof of Theorem 8 yield the existence of solutions (in the sense of Definition 6) also when (23) is supplemented with a forcing term $F \in L^2(0,T;V')$ on the right-hand side.

**Generalized semiflow and long-time behavior of the weak solutions.** The long-time dynamics of the weak solutions to (6) will be analyzed in the phase space $\mathcal{X}_w := L^4(\Omega)$ (endowed with the metric of $\|\cdot\|_{L^4(\Omega)}$).

**Theorem 10:** Assume (13)-(14), (21), and (31). Then, the set of the weak solutions $\mathcal{S}_w$ is a generalized semiflow on $\mathcal{X}_w$, enjoying the continuity properties **(C3)** and **(C4)**.

**Theorem 11:** Assume (13)-(14), (21), and (31). Then, the generalized semiflow $\mathcal{S}_w$ possesses a unique global attractor $\mathcal{A}_w$ on $\mathcal{X}_w$, given by $\mathcal{A}_w := \omega(\mathcal{X}_w)$. Moreover, $\mathcal{A}_w$ is Lyapunov stable.

Henceforth, we adopt the convention of denoting by the same symbol $C$, whose meaning may vary even within the same line, (almost) all the constants occurring in the estimates.

## 4. Generalized semiflow and Global Attractor of the *weak* solutions

### 4.1. *Proof of Theorem 8.*

We fix a sequence $\{\chi_0^k\}_k \subset V$ (which we shall suppose to suitably approximate the initial datum $\chi_0$ of Problem 1), and for any $k \in \mathbb{N}$ we consider the following Cauchy problem:

**Problem $\mathbf{P_k}$:** *Find* $\chi_k \in H^1(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;W)$ *and* $u_k \in L^2(0,T;V)$ *fulfilling* (23), $\chi(0) = \chi_0^k$, *and*

$$\partial_t \chi_k + J\chi_k + \chi_k^3 - \chi_k = \rho(u_k) \quad \text{in } H, \quad \text{for a.e. } t \in (0,T). \tag{34}$$

A straightforward adaptation of the argument in the proof of Thm. 2.2 in [19] ensures that Problem $\mathbf{P_k}$ admits a unique solution $(\chi_k, u_k)$ for any $k \in \mathbb{N}$.

The following result yields our existence Theorem 8.

**Proposition 12:** *Under the assumptions* (13)-(15), (21), (31)-(32), *suppose that* $\{\chi_0^k\}_k \subset V$ *, and*

$$\chi_0^k \to \chi_0 \quad \text{in } L^4(\Omega). \quad \text{as } k \uparrow \infty. \tag{35}$$

*Let* $\{(\chi_k, u_k)\}$ *be the sequence of the solutions to* $\mathbf{P_k}$*, supplemented with the data* $\{\chi_0^k\}$*.*
*Then, there exist* $\chi \in H^1(0,T;V') \cap C^0([0,T];H) \cap L^2(0,T;V) \cap C^0_w([0,T];L^4(\Omega)) \cap L^6(0,T;L^6(\Omega))$*, and* $u \in L^2(0,T;V)$*, such that, up to the extraction of a subsequence, the following convergences hold as* $k \uparrow \infty$

$$\chi_k \rightharpoonup^* \chi \quad \text{in } H^1(0,T;V') \cap L^\infty(0,T;L^4(\Omega)) \cap L^2(0,T;V), \tag{36}$$

$$\chi_k \to \chi \quad \text{in } C^0([0,T];V') \cap L^2(0,T;L^{6-\varepsilon}(\Omega)) \cap L^p(0,T;L^4(\Omega)) \quad \text{for any } 0 < \varepsilon \le 5 \text{ and } 1 \le p < \infty, \tag{37}$$

$$\begin{cases} \chi_k(t) \to \chi(t) & \text{in } L^4(\Omega)) \quad \text{for a.e. } t \in (0,T), \\ \chi_k(t) \rightharpoonup \chi(t) & \text{in } L^4(\Omega)) \quad \text{for all } t \in [0,T], \end{cases} \tag{38}$$

$$u_k \rightharpoonup u \quad \text{in } L^2(0,T;V), \tag{39}$$

*and* $(\chi, u)$ *is a solution to Problem 1. Moreover, the pair* $(\chi, u)$ *fulfils* (30), *and* $\chi$ *has the additional regularity* (33).

**Proof:** First, we provide *a priori* estimates for the norms of the approximate solutions $\{(\chi_k, u_k)\}$ in suitable function spaces.

***A priori* estimates.** Let us test (23) by $J^{-1}(\rho(u_k))$, (34) by $J^{-1}(\partial_t \chi_k) + \chi_k^3$, add the resulting equations and integrate on $(0,t)$, for $t \in (0,T]$. Two terms cancel out; taking into account (19) and (20) (which for example yields

$$\langle J\chi_k(t), J^{-1}(\partial_t \chi_k(t))\rangle = (\partial_t \chi_k(t), \chi_k(t))_H = \frac{1}{2}\frac{d}{dt}\|\chi_k\|_H^2(t)$$

for a.e. $t \in (0,T)$), we obtain

$$\begin{aligned}\int_0^t (u_k(s), \rho(u_k(s)))_H \, ds + \int_0^t \|\partial_t \chi_k(s)\|_{V'}^2 ds + \int_0^t \langle \chi_k^3(s), J^{-1}(\partial_t \chi_k(s))\rangle \, ds \\ + \frac{1}{2}\|\chi_k(t)\|_H^2 + \frac{1}{4}\|\chi_k(t)\|_{L^4(\Omega)}^4 + \int_0^t \langle J\chi_k(s), \chi_k^3(s)\rangle \, ds + \int_0^t \|\chi_k^3(s)\|_H^2 ds \\ = \frac{1}{2}\|\chi_0^k\|_H^2 + \frac{1}{4}\|\chi_0^k\|_{L^4(\Omega)}^4 \\ + \int_0^t \langle \chi_k(s), J^{-1}(\partial_t \chi_k(s))\rangle \, ds + \int_0^t \|\chi_k(s)\|_{L^4(\Omega)}^4 \, ds \\ + \int_0^t (\rho(u_k(s)), \chi_k^3(s))_H ds.\end{aligned} \tag{40}$$

Note that, in view of (21),

$$\int_0^t (u_k(s), \rho(u_k(s)))_H \, ds \geq m \int_0^t \|\rho(u_k(s))\|_H^2 ds,$$

while the sixth term on the left-hand side of (40) is positive thanks to (15).

On the other hand, by (31) we can fix a constant $\varepsilon_1 > 0$ such that $m > 1/3 + \varepsilon_1$, which guarantees, by elementary computations, that there exists a constant $\sigma_1 > 1/4$ such that

$$1 - \frac{1}{4(m-\varepsilon_1)} > \sigma_1.$$

Hence, we combine

$$\begin{aligned}\left|\int_0^t (\rho(u_k(s)), \chi_k^3(s))_H \, ds\right| \leq (m-\varepsilon_1)\int_0^t \|\rho(u_k(s))\|_H^2 \, ds \\ + \frac{1}{4(m-\varepsilon_1)}\int_0^t \|\chi_k^3(s)\|_H^2 \, ds,\end{aligned}$$

with the following estimates:

$$\left|\int_0^t \langle \chi_k^3(s), J^{-1}(\partial_t \chi_k(s))\rangle \, ds\right| \le \sigma_1 \int_0^t \|\chi_k^3(s)\|_{V'}^2 \, ds + \frac{1}{4\sigma_1}\int_0^t \|J^{-1}(\partial_t \chi_k(s))\|_V^2 ds \le \sigma_1 \int_0^t \|\chi_k^3(s)\|_H^2 \, ds + \frac{1}{4\sigma_1}\int_0^t \|\partial_t \chi_k(s)\|_{V'}^2 \, ds,$$

and

$$\left|\int_0^t \langle \chi_k(s), J^{-1}(\partial_t \chi_k(s))\rangle \, ds\right| \le \kappa_1 \int_0^t \|\partial_t \chi_k(s)\|_{V'}^2 \, ds + \frac{1}{4\kappa_1}\int_0^t \|\chi_k(s)\|_H^2 \, ds,$$

where we have chosen $0 < \kappa_1 < 1 - 1/(4\sigma_1)$ (which is possible since $\sigma_1 > 1/4$).

We thus deduce from (40) and the above estimates that there exist three constants $c_1, c_2, c_3 > 0$ such that

$$\begin{aligned} c_1 \int_0^t \|\rho(u_k(s))\|_H^2 \, ds + c_2 \int_0^t \|\partial_t \chi_k(s)\|_{V'}^2 \, ds + c_3 \int_0^t \|\chi_k^3(s)\|_H^2 \, ds \\ + \frac{1}{2}\|\chi_k(t)\|_H^2 + \frac{1}{4}\|\chi_k(t)\|_{L^4(\Omega)}^4 \le \frac{1}{2}\|\chi_0^k\|_H^2 + \frac{1}{4}\|\chi_0^k\|_{L^4(\Omega)}^4 \\ + \frac{1}{4\kappa_1}\int_0^t \|\chi_k(s)\|_H^2 \, ds + \int_0^t \|\chi_k(s)\|_{L^4(\Omega)}^4 \, ds. \end{aligned} \tag{41}$$

Recalling (35), an easy application of the Gronwall's Lemma yields that there exists a constant $C \ge 0$, independent of $k \in \mathbb{N}$, such that

$$\|\chi_k\|_{L^\infty(0,T;L^4(\Omega))\cap H^1(0,T;V')} + \|\chi_k^3\|_{L^2(0,T;H)} + \|\rho(u_k)\|_{L^2(0,T;H)} \le C. \tag{42}$$

By comparison in (23), we get $\|J(u_k)\|_{L^2(0,T;V')} \le C$, hence

$$\|u_k\|_{L^2(0,T;V)} + \|\rho(u_k)\|_{L^2(0,T;H^1(\Omega))} \le C \quad \forall k \in \mathbb{N}, \tag{43}$$

the second estimate following from the Lipschitz continuity of $\rho$. Finally, testing (34) by $\chi_k$ and integrating on the interval $(0,t)$, we find

$$\frac{1}{2}\|\chi_k(t)\|_H^2 + \int_0^t \langle J(\chi_k(s)), \chi_k(s)\rangle \, ds + \int_0^t \|\chi_k(s)\|_{L^4(\Omega)}^4 \le \frac{1}{2}\|\chi_0^k\|_H^2 + \frac{1}{2}\int_0^t \|\rho(u_k(s))\|_{L^2(0,T;H)}^2 \, ds + \frac{3}{2}\int_0^t \|\chi_k(s)\|_H^2 \, ds.$$

In view of (19) and (42)-(43), this entails

$$\|\chi_k\|_{L^2(0,T;V)} \leq C \quad \forall k \in \mathbb{N}. \tag{44}$$

**Compactness for the approximate solutions.** The *a priori estimates* (42) and (44) and the Lions-Aubin's theorem (see Thm.4, Cor.5 in [23]) yield that $\{\chi_k\}$ is relatively compact for the strong topologies of $C^0([0,T];V')$ and of $L^2(0,T;L^{6-\varepsilon}(\Omega))$, for any $0 < \varepsilon \leq 5$. Note that the latter compactness property is due to the continuity of the embedding $V \subset H^1(\Omega)$, and to the compactness of $H^1(\Omega) \subset\subset L^{6-\varepsilon}(\Omega)$. On the other hand, $\{\chi_k\}$ is relatively weakly (weakly star) compact in $H^1(0,T;V') \cap L^\infty(0,T;L^4(\Omega)) \cap L^2(0,T;V)$. Analogously, thanks to (43) $\{u_k\}$ and $\{\rho(u_k)\}$ are relatively weakly compact in $L^2(0,T;V)$, and so is $\{\chi_k^3\}$ in $L^2(0,T;H)$.

Thus, there exists a subsequence, which we do not relabel, and a triplet $(\chi, u, \zeta)$, such that $u \in L^2(0,T;V)$, $\zeta \in L^2(0,T;H^1(\Omega))$, $\chi \in \big(H^1(0,T;V') \cap L^2(0,T;V) \cap L^\infty(0,T;L^4(\Omega))\big) \subset C^0([0,T];H) \cap C^0_w([0,T];L^4(\Omega))$ (the last inclusion following from Lemma III.1.4 in [24]), and the convergences (36), (37), and (39) hold. Combining the first of (37) with (35), we have $\chi(0) = \chi_0$. Besides,

$$\rho(u_k) \rightharpoonup \zeta \quad \text{in } L^2(0,T;H^1(\Omega)) \quad \text{as } k \uparrow \infty. \tag{45}$$

As for (38), the a.e. pointwise convergence in $L^4(\Omega)$ of $\chi_k$ is, upon the extraction of a subsequence, a consequence of the strong convergence of $\chi_k$ to $\chi$ in $L^2(0,T;L^4(\Omega))$. Note that this pointwise convergence and the bound for $\chi_k$ in $L^\infty(0,T;L^4(\Omega)$ imply the last of (37) via the dominated convergence theorem. On the other hand, the second of (38) can be proved by considering on the separable reflexive space $L^4(\Omega)$ the norm $|||\cdot|||$ which induces the weak convergence on the bounded subsets of $L^4(\Omega)$ (cf. e.g. Chap. III in [5] for the definition of $|||\cdot|||$). Hence, the estimates (42) and an argument of Ascoli-Arzelà type (cf. Cor. 5 in [23]) yield (38). Along the same subsequence, we also have

$$\begin{gathered} \chi_k^3 \to \chi^3 \quad \text{in } L^2(0,T;V'), \\ \text{whence } \chi^3 \in L^2(0,T;H), \text{ and } \quad \chi_k^3 \rightharpoonup \chi^3 \quad \text{in } L^2(0,T;H). \end{gathered} \tag{46}$$

(the latter conclusion follows from the fact that $\{\chi_k^3\}$ is weakly relatively compact in $L^2(0,T;H)$). In particular, $\chi \in L^6(0,T;L^6(\Omega))$. Indeed, to check (46), let us fix a constant $0 < \tau < 1/2$: then, elementary computations and Hölder's inequality yield that there exists a constant $0 < \varepsilon_\tau \leq 5$

such that for a.e. $t \in (0,T)$

$$
\begin{aligned}
&\left|\langle \chi_k^3(t) - \chi^3(t), v\rangle\right| \\
&\le \|v\|_{L^6(\Omega)} \|\chi_k(t) - \chi(t)\|_{L^{6-\varepsilon_\tau}(\Omega)} \|\chi_k^2(t) + \chi^2(t)\|_{L^{\tau+3/2}(\Omega)} \quad \forall v \in V.
\end{aligned}
$$

Hence, we easily obtain

$$
\begin{aligned}
&\int_0^T \|\chi_k^3(t) - \chi^3(t)\|_{V'}^2 \, dt \\
&\le \int_0^T \|\chi_k(t) - \chi(t)\|_{L^{6-\varepsilon_\tau}(\Omega)}^2 \left( \|\chi_k^2(t)\|_{L^{\tau+3/2}(\Omega)}^2 + \|\chi^2(t)\|_{L^{\tau+3/2}(\Omega)}^2 \right) dt \\
&\le \int_0^T \|\chi_k(t) - \chi(t)\|_{L^{6-\varepsilon_\tau}(\Omega)}^2 \left( \|\chi_k(t)\|_{L^{3+2\tau}(\Omega)}^4 + \|\chi(t)\|_{L^{3+2\tau}(\Omega)}^4 \right) dt \\
&\le \left( \|\chi_k\|_{L^\infty(0,T;L^4(\Omega))}^4 + \|\chi\|_{L^\infty(0,T;L^4(\Omega))}^4 \right) \int_0^T \|\chi_k(t) - \chi(t)\|_{L^{6-\varepsilon_\tau}(\Omega)}^2 \, dt,
\end{aligned}
$$

which yields (46) thanks to (37) and (42).

**Passage to the limit and proof of existence.** It follows from (35) and (37) that $\chi(0) = \chi_0$; moreover, (36) and (39) also yield that $(\chi, u)$ solves (23). On the other hand, the convergences (36)-(37) and (45)-(46) allow to pass to the limit in (34), thus obtaining

$$
\partial_t \chi + J\chi + \chi^3 - \chi = \zeta \quad \text{in } V', \quad \text{for a.e. } t \in (0,T). \tag{47}
$$

Hence, to conclude that $(\chi, u)$ is a solution to Problem 1, it is sufficient to prove that $\zeta = \rho(u)$. Note that $\rho$ induces a maximal monotone graph on $L^2(0,T;H)$: by the theory of maximal monotone operators (cf. Prop.1.1, p.42, in [4]), it is then sufficient to show that

$$
\limsup_{k\uparrow\infty} \int_0^t (\rho(u_k(s)), u_k(s))_H \, ds \le \int_0^t (\zeta(s), u_k(s))_H \, ds. \tag{48}
$$

In fact, testing (23) by $J^{-1}(\rho(u_k))$ and (34) by $J^{-1}(\partial_t \chi_k)$, we obtain

$$
\begin{aligned}
&\limsup_{k\uparrow\infty} \int_0^t (\rho(u_k(s)), u_k(s))_H \, ds \\
&= \limsup_{k\uparrow\infty} \left( - \int_0^t \|\partial_t \chi_k(s)\|_{V'}^2 \, ds - \frac{1}{2}\|\chi_k(t)\|_H^2 \right) + \frac{1}{2} \lim_{k\uparrow\infty} \|\chi_0^k\|_H^2 \\
&- \lim_{k\uparrow\infty} \int_0^t \left( \langle \chi_k^3(s), J^{-1}(\partial_t \chi_k(s))\rangle - \langle \chi_k(s), J^{-1}(\partial_t \chi_k(s))\rangle \right) \, ds \\
&\leq - \int_0^t \|\partial_t \chi(s)\|_{V'}^2 \, ds - \frac{1}{2}\|\chi(t)\|_H^2 + \frac{1}{2}\|\chi_0\|_H^2 \\
&- \int_0^t \left( \langle \chi^3(s), J^{-1}(\partial_t \chi(s))\rangle - \langle \chi(s), J^{-1}(\partial_t \chi(s))\rangle \right) \, ds \\
&= - \int_0^t \langle \zeta(s), J^{-1}(\partial_t \chi_k(s))\rangle = \int_0^t \langle J(u(s)), J^{-1}(\zeta(s))\rangle \, ds,
\end{aligned}
\tag{49}
$$

whence (48) by (20). Indeed, the intermediate inequality in (49) follows from (35) (note that this is the first point where we use the strong convergence of $\chi_0^k$ to $\chi_0$ in $H$), and from the weak and strong convergences (36)-(38), also combined with (46). In particular, (38) yields

$$
\liminf_{k\uparrow\infty} \|\chi_k(t)\|_H^2 \geq \|\chi(t)\|_H^2 \quad \forall t \in [0,T].
$$

The final identities in (49) are due to (47) and to (23). Thus, (45) becomes

$$
\rho(u_k) \rightharpoonup \rho(u) \quad \text{in } L^2(0,T;V) \text{ as } k \uparrow \infty. \tag{50}
$$

In the end, let us point out that, since $\chi$ fulfils (24) and $\rho(u) \in L^2(0,T;H)$, by standard regularity results for parabolic equations (see e.g. [25, Chap.3]), $\chi$ has the further regularity

$$
\chi \in C^0([\delta,T]; L^4(\Omega)) \quad \forall\, \delta > 0. \tag{51}
$$

**Proof of** (30) **and of** (33)**.** Due to (35), $\chi_k$ converges to $\chi$ in $L^4(\Omega)$ for $t = 0$; in general, by (38), there exists a negligible set $\overline{\mathcal{N}} \subset (0,T]$ outside which $\chi_k$ strongly converges pointwisely to $\chi$ in $L^4(\Omega)$. Hence, let us prove (30) for any $t \in (0,T]$ and $s \in (0,t] \setminus \overline{\mathcal{N}}$. Indeed, by (35), (38) and the weak lower semicontinuity of the norm, for any such $t$ and $s$ we have

$$
\liminf_{k\uparrow\infty} \|\chi_k(t)\|_{L^4(\Omega)}^4 \geq \frac{1}{4}\|\chi(t)\|_{L^4(\Omega)}^4, \quad \chi_k(s) \to \chi(s) \quad \text{in } L^4(\Omega). \tag{52}
$$

Let us then test (34) by $\chi_k^3$ and integrate on the interval $(s,t)$. Taking the $\liminf_{k\uparrow\infty}$ of the resulting relation and developing the same computations as for (40), we obtain

$$0 \geq \frac{1}{4} \liminf_{k\uparrow\infty} \|\chi_k(t)\|^4_{L^4(\Omega)} - \frac{1}{4} \lim_{k\uparrow\infty} \|\chi_k(s)\|^4_{L^4(\Omega)}$$
$$+ \liminf_{k\uparrow\infty} \int_s^t \langle J\chi_k(\tau), \chi_k^3(\tau)\rangle d\tau + \liminf_{k\uparrow\infty} \int_s^t \Big( \|\chi_k^3(\tau)\|^2_H - \|\chi_k(\tau)\|^4_{L^4(\Omega)} \Big) d\tau$$
$$- \lim_{k\uparrow\infty} \int_s^t (\rho(u_k(\tau)), \chi_k^3(\tau))_H d\tau \geq \frac{1}{4}\|\chi(t)\|^4_{L^4(\Omega)} - \frac{1}{4}\|\chi(s)\|^4_{L^4(\Omega)}$$
$$+ \int_s^t \Big( \|\chi^3(\tau)\|^2_H + \|\chi(\tau)\|^4_{L^4(\Omega)} - (\rho(u(\tau)), \chi^3(\tau))_H \Big)\ d\tau, \tag{53}$$

which trivially yields (30). Note that (53) follows from (52), (15), (37), and by combining the strong convergence (46) and the weak one (50).

Owing to (51), in order to conclude (33) it is sufficient to show that

$$\forall \{t_n\} \subset (0,T] \text{ with } t_n \downarrow 0, \quad \chi(t_n) \to \chi_0 \text{ in } L^4(\Omega) \text{ as } n\uparrow\infty. \tag{54}$$

By (27), we have $\liminf_{n\uparrow\infty} \|\chi(t_n)\|^4_{L^4(\Omega)} \geq \|\chi_0\|^4_{L^4(\Omega)}$. On the other hand, (30) reads, for $t = t_n$ and $s = 0$,

$$\frac{1}{4}\|\chi(t_n)\|^4_{L^4(\Omega)} + \int_0^{t_n} \Big( \|\chi^3(\tau)\|^2_H - \|\chi(\tau)\|^4_{L^4(\Omega)} - \big(\rho(u(\tau)), \chi^3(\tau)\big)\big)_H \Big)\ d\tau$$
$$\leq \frac{1}{4}\|\chi_0\|^4_{L^4(\Omega)}. \tag{55}$$

Taking the $\limsup$ as $n\uparrow\infty$ of (55), we deduce $\limsup_{n\uparrow\infty} \|\chi(t_n)\|^4_{L^4(\Omega)} \leq \|\chi_0\|^4_{L^4(\Omega)}$, whence (54). ☐

**Remark 13:** Let us stress that, while the continuity on $[\delta, T]$, for all $\delta > 0$, of the above weak solution $\chi$ is a consequence of the structure of equation (24), the continuity in $t = 0$ is obtained by combining the weak continuity $\chi \in C^0_w([0,T]; L^4(\Omega))$ with the energy inequality (30). As a consequence, we conclude that

$$\text{any } \chi \in \mathcal{S}_w \text{ has the regularity} \quad \chi \in C^0([0,T]; L^4(\Omega))\ \forall\, T > 0. \tag{56}$$

### 4.2. *Proof of Theorem 10*

**Proof:** It follows from Theorem 8 that for any $\chi_0 \in \mathcal{X}_w$ there exists a weak solution $\chi \in \mathcal{S}_w$ fulfilling $\chi(0) = \chi_0$, so that $\mathcal{S}_w$ complies with **(H1)**; **(C3)** holds thanks to Remark 13.

It can be readily checked that $\mathcal{S}_w$ fulfils **(H3)**; further, in view of (56) and of (30), the map $t \mapsto \mathcal{V}(u,\chi)(t)$ is continuous and non-increasing, so that **(H2)** holds.

We shall verify **(C4)**, which trivially entails **(H4)**. To this aim, we fix a sequence

$$\chi_0^n \text{ converging to } \chi^0 \text{ in } L^4(\Omega), \tag{57}$$

and a sequence $\{(\chi_n, u_n)\} \subset \mathcal{S}_w$ of weak solutions fulfilling $\chi_n(0) = \chi_0^n$. Like in the proof of Proposition 12, we test (28) by $J^{-1}(\rho(u_n))$, (29) by $J^{-1}(\partial_t \chi_n) + \chi_n^3$, integrate on $(0,t)$ for any $t > 0$, and add the resulting relations. Hence, we obtain the estimates (42)-(44) for the sequence $\{(\chi_n, u_n)\}$. Note that, on this level, testing (29) by $\chi_n^3$ is only a formal estimate, since the regularity (27) for $(\chi_n, u_n)$ does not guarantee $\chi_n^3 \in V$. Nonetheless, $\chi_n^3 \in H$, and, by comparison in (29), we see that $\partial_t \chi_n + J\chi_n$ is in $H$ as well. Replacing the test function $\chi_n^3$ by its Yosida regularization, we can thus make the whole procedure rigorous.

Arguing exactly as in the same way as in the proof of Proposition 12, we conclude from the estimates (42)-(44) that there exists a limit pair $(\chi, u)$ with the regularity (27), and a subsequence $\{(\chi_{n_k}, u_{n_k})\}$ (extracted by a diagonal argument), along which the convergences (36)-(39) and (45)-(46) hold for any $T > 0$. This entails that $\chi(0) = \chi^0$, and that $(\chi, u)$ fulfils (28) and (29). Moreover, since $(\chi_{n_k}, u_{n_k})$ complies with (30), passing to the limit we conclude that $(\chi, u)$ fulfils (30) as well (cf. with (53)), and it is thus a weak solution in the sense of Definition 6.

Therefore, thanks to (56), the map $t \mapsto \mathcal{V}(\chi, u)(t)$ is continuous on $[0, +\infty)$ and non increasing on $(0, +\infty)$.

In view of (57) and of the pointwise weak convergence (38), **(H4)** follows once we prove that

$$\limsup_{k \uparrow \infty} \|\chi_{n_k}(t)\|_{L^4(\Omega)} \le \|\chi(t)\|_{L^4(\Omega)} \quad \forall\, t > 0, \tag{58}$$

which can be obtained arguing in the same way as in the proof of Prop.7.4 in [1]. Namely, the convergences (36)-(38) and (46) and (50) yield

$$\mathcal{V}(\chi_{n_k}(t), u_{n_k}(t)) \to \mathcal{V}(\chi(t), u(t)) \quad \text{for a.e. } t \in [0, +\infty).$$

However, since both maps $t \mapsto \mathcal{V}(\chi_{n_k}(t), u_{n_k}(t))$ and $t \mapsto \mathcal{V}(\chi(t), u(t))$ are continuous and non increasing on $[0, +\infty)$ thanks to Remark 13, the latter convergence necessarily holds *for all* $t \in [0, +\infty)$, i.e.

$$\int_0^t \|\chi_{n_k}^3(\tau)\|_H^2 - \|\chi_{n_k}(\tau)\|_{L^4(\Omega)}^4 - \big(\rho(u_{n_k}(\tau)), \chi_{n_k}^3(\tau)\big)_H \; d\tau + \frac{1}{4}\|\chi_{n_k}(t)\|_{L^4(\Omega)}^4$$
$$\downarrow$$
$$\int_0^t \|\chi^3(\tau)\|_H^2 - \|\chi(\tau)\|_{L^4(\Omega)}^4 - \big(\rho(u(\tau)), \chi^3(\tau)\big)_H \; d\tau + \frac{1}{4}\|\chi(t)\|_{L^4(\Omega)}^4 \tag{59}$$

$\forall\, t \in [0, +\infty)$. By a lower semicontinuity argument, (59) yields (58), as well as $\limsup_{k\uparrow\infty} \int_0^t \|\chi_{n_k}^3(\tau)\|_H^2 d\tau \le \int_0^t \|\chi^3(\tau)\|_H^2 d\tau$, so that

$$\chi_{n_k}^3 \to \chi^3 \quad \textit{strongly} \text{ in } L^2(0, T; H). \tag{60}$$

Finally, (59), combined with (60) and (50) also entails that

$$\chi_{n_k}(t) \to \chi(t) \quad \text{ in } L^4(\Omega) \quad \text{uniformly on the compact subsets of } [0, +\infty),$$

which gives **(C4)**. □

**Remark 14:** Indeed, arguing along the same lines as in the proof of (33) in Theorem 8 and of the property continuity **(C4)**, it is possible to obtain the following characterization, which is the analogue of Prop. 7.4 in [1]. Namely, that the following conditions are equivalent:

***i).*** $\mathcal{S}_w$ is generalized semiflow on $\mathcal{X}_w$;
***ii).*** each $\chi \in \mathcal{S}_w$ is continuous from $(0, +\infty)$ to $\mathcal{X}_w$;
***iii).*** each $\chi \in \mathcal{S}_w$ is continuous from $[0, +\infty)$ to $\mathcal{X}_w$.

### 4.3. *Proof of Theorem 11*

**Proof:** We shall check that $\mathcal{S}_w$ complies with the two necessary and sufficient (by Theorem 3) conditions for the existence of the global attractor: namely, that

$$\mathcal{S}_w \text{ is asymptotically compact,} \tag{61}$$
$$\mathcal{S}_w \text{ is point dissipative.} \tag{62}$$

The Lyapunov stability of the attractor $\mathcal{A}_w$ shall then follow from the continuity properties **(C3)** and **(C4)** of $\mathcal{S}_w$.

**Ad** (62)**.** (62) ensues from suitable *a priori* estimates on the weak solutions. Indeed, let us fix an element $(\chi, u) \in \mathcal{S}_w$, and let us test (28) by $J^{-1}(\rho(u))$, (29) by $J^{-1}(\partial\chi_t) + \chi^3$, and add the resulting relations. Thus, developing the same computations as for concluding (41) in the proof of Proposition 12, we deduce that for a.e. $t \in (0,+\infty)$

$$\begin{aligned} c_1\|\rho(u(t))\|_H^2 + c_2\|\partial_t\chi(t)\|_{V'}^2 + c_3\|\chi^3(t)\|_H^2 + \frac{1}{2}\frac{d}{dt}\|\chi\|_H^2(t) \\ +\frac{1}{4}\frac{d}{dt}\|\chi\|_{L^4(\Omega)}^4(t) \le \frac{1}{4\kappa_1}\|\chi(t)\|_H^2 + \|\chi(t)\|_{L^4(\Omega)}^4, \end{aligned} \tag{63}$$

with $c_1, c_2, c_3 > 0$ and $0 < \kappa_1 < 1$ as in (41), *independent* of $t$. Again, note that, on this level, such computations are only formal: on the other hand, they could be made rigorous by performing the regularization procedure hinted in the proof of Theorem 10.

Next, we test (29) by $\chi$. Also taking into account (19), we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\chi\|_H^2(t) + \|\chi(t)\|_V^2 + \|\chi(t)\|_{L^4(\Omega)}^4 \\ &\le \frac{c_1}{4}\|\rho(u(t))\|_H^2 + \left(\frac{1}{c_1}+1\right)\|\chi(t)\|_H^2. \end{aligned} \tag{64}$$

We multiply (64) by 2 and add it to (63): easy computations entail

$$\begin{aligned} \frac{3}{2}\frac{d}{dt}\|\chi\|_H^2(t) + \frac{1}{4}\frac{d}{dt}\|\chi\|_{L^4(\Omega)}^4(t) + \|\chi(t)\|_{L^4(\Omega)}^4 &\le \left(\frac{2}{c_1}+2+\frac{1}{4\kappa_1}\right)\|\chi(t)\|_H^2 \\ &\le \frac{3}{4}\|\chi(t)\|_{L^4(\Omega)}^4 + K_1 - \frac{3}{2}\|\chi(t)\|_H^2, \end{aligned}$$

where we have also used Hölder's inequality, yielding that for any constants $r, R > 0$ there exists a constant $K = K(r, R, |\Omega|)$ such that $R\|v\|_H^2 \le r\|v\|_{L^4(\Omega)}^4 + K$ for any $v \in L^4(\Omega)$. Therefore, a differential form of the Gronwall Lemma (see e.g. Lemma 2.5 in [10]) gives for all $t > 0$

$$\Phi(\chi(t)) \le \Phi(\chi(0))\exp(-t) + K_1, \quad \Phi(\chi) := \frac{3}{2}\|\chi\|_H^2 + \frac{1}{4}\|\chi\|_{L^4(\Omega)}^4. \tag{65}$$

Defining

$$\mathcal{B}_0 := \{\chi \in \mathcal{X}_w \,:\, \|\chi\|_{L^4(\Omega)} \le 2K_1\}$$

we deduce from (65) that for any weak solution $(\chi, u)$ there exists $\tau > 0$, depending on $K_1$ and $\chi(0)$, such that $\chi(t) \in \mathcal{B}_0$ for $t \ge \tau$.

**Ad** (61)**.** It can be readily checked by (65) that $\mathcal{S}_w$ is eventually bounded. Owing to Remark 2, (61) follows once we prove that $\mathcal{S}_w$ is compact. To this aim, we fix a bounded sequence $\{\chi_0^n\} \subset L^4(\Omega)$, and denote by $\{(\chi_n, u_n)\}$ the sequence of the associated weak solutions. We may suppose that $\chi_0^n$ weakly converges, possibly on a subsequence, to some $\chi_\infty^0$ in $L^4(\Omega)$. Exactly as in the proof of the upper-semicontinuity property **(H4)** (cf. the proof of Theorem 10), upon regularizing we may repeat the estimates in the proof of Proposition 12 (which in fact only rely on the *boundedness* of the approximate initial data in $L^4(\Omega)$).

Thus, there exists a limit triplet $(\chi_\infty, u_\infty, \zeta_\infty)$, with $(\chi_\infty, u_\infty)$ as in (27) and $\zeta_\infty \in L^2(0,T;V)$, and a diagonal subsequence (which we do not relabel) of $\{(\chi_n, u_n)\}$, along which the convergences (36)-(39) and (45)-(46) hold for any $T > 0$. We aim at proving that, up to the extraction of a subsequence,

$$\chi_n(t) \to \chi_\infty(t) \quad \text{in } L^4(\Omega) \quad \forall t > 0. \tag{66}$$

Again, $\chi_\infty(0) = \chi_\infty^0$, the pair $(\chi_\infty, u_\infty)$ complies with (28). However, we cannot conclude anymore that $(\chi_\infty, u_\infty)$ fulfils (29) via (49), due to the only *weak* convergence of $\{\chi_0^n\}$: indeed, we just conclude that

$$\partial_t \chi_\infty + J\chi_\infty + \chi_\infty^3 - \chi_\infty = \zeta_\infty \quad \text{a.e. in } (0,+\infty). \tag{67}$$

The next step consists in showing that

$$\chi_n(t) \to \chi_\infty(t) \quad \text{in } H \quad \forall t > 0. \tag{68}$$

Indeed, (38) guarantees that for any $t > 0$, there exists $0 < \tau < t$ such that

$$\chi_n(\tau) \to \chi_\infty(\tau) \quad \text{in } H. \tag{69}$$

Hence, testing (29) (written for $\chi_n$) by $\chi_n$ and integrating on the interval $(\tau, t)$, we get

$$\begin{aligned} \limsup_{n\uparrow\infty} \frac{1}{2}\|\chi_n(t)\|_H^2 &\le \limsup_{n\uparrow\infty} \frac{1}{2}\|\chi_n(\tau)\|_H^2 \\ &+ \int_\tau^t \Big( -\|\chi_n(r)\|_V^2 - \|\chi_n(r)\|_{L^4(\Omega)}^4 + \|\chi_n(r)\|_H^2 + \langle \rho(u_n(r)), \chi_n(r)\rangle \Big)\, dr \\ &\le \frac{1}{2}\|\chi_\infty(\tau)\|_H^2 - \int_\tau^t \Big( \|\chi_\infty(r)\|_V^2 + \|\chi_\infty(r)\|_{L^4(\Omega)}^4 - \|\chi_\infty(r)\|_H^2 \Big)\, dr \\ &+ \int_\tau^t \langle \zeta_\infty(r), \chi_\infty(r)\rangle\, dr = \frac{1}{2}\|\chi_\infty(t)\|_H^2 \end{aligned} \tag{70}$$

where the second passage follows from the convergences (36)-(37), (45), and (69), whereas the final equality is a consequence of (67). In view of the pointwise weak convergence (38), we deduce (68) from (70).

Now, we test (28) by $J^{-1}(\rho(u_n))$, (29) for $J^{-1}(\partial_t(\chi_n))$, add the resulting equations and integrate on $(s,t)$, for *any* $0 < s \le t < +\infty$. Thanks to (68), we can repeat the estimate (49) and get that

$$\limsup_{n\uparrow\infty} \int_s^t (\rho(u_n)(r), u_n(r))_H\, dr \le \int_s^t (\zeta_\infty(r), u_\infty(r))_H\, dr \quad \forall\, 0 < s \le t < \infty,$$

which yields that $\zeta_\infty(t) = \rho(u_\infty(t))$ for a.e. $t \in (0,+\infty)$, and the pair $(\chi_\infty, u_\infty)$ also fulfils (29).

Arguing as in the proof of Theorem 10, we manage to pass to the limit in (30) for $(\chi_n, u_n)$: thus, there exists a negligible set $\mathcal{N} \subset [0,+\infty)$ such that $(\chi_\infty, u_\infty)$ complies with (30) for all $t \in (0,+\infty)$ and for $s \in (0,t] \setminus \mathcal{N}$. Therefore, $\chi$ is a weak solution in the sense of Definition 6 on any half-line $[\delta,+\infty)$, $\delta > 0$. Arguing as in the proof of Theorem 10, we thus conclude that

$$\chi_n(t) \to \chi(t) \quad \text{in } L^4(\Omega) \quad \forall\, t \geq \delta,\ \forall\, \delta > 0,$$

whence (66). □

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