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Well-posedness and asymptotic analysis for a Penrose–Fife type phase field system

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SUMMARY

In this paper, an asymptotic analysis of the (non-conserved) Penrose–Fife phase field system for two vanishing time relaxation parameters ε and δ is developed, in analogy with the similar analyses for the phase field model proposed by G. Caginalp (*Arch. Rational Mech. Anal.* 1986; **92**:205–245), which were carried out by Rossi and Stoth (*Adv. Math. Sci. Appl.* 2003; **13**:249–271; *Quart. Appl. Math.* 1995; **53**:695–700).

Although formally the singular limits for $\varepsilon \downarrow 0$ and for ε and $\delta \downarrow 0$ are, respectively, the viscous Cahn–Hilliard equation and the Cahn–Hilliard equation, it turns out that the Penrose–Fife system is indeed a *bad* approximation for these equations. Therefore, we consider an *alternative* approximating phase field system, which could be viewed as a generalization of the classical Penrose–Fife phase field system, featuring a double non-linearity given by two maximal monotone graphs. A well-posedness result is proved for such a system, and it is shown that the solutions converge to the unique solution of the viscous Cahn–Hilliard equation as $\varepsilon \downarrow 0$, and of the Cahn–Hilliard equation as $\varepsilon \downarrow 0$ and $\delta \downarrow 0$.
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KEY WORDS: Penrose–Fife phase field model; Cahn–Hilliard equation; asymptotic analysis

1. INTRODUCTION

This paper is concerned with the asymptotic analysis in terms of two vanishing time relaxation parameters of the phase field model proposed by Penrose and Fife [1,2] for the modelling of the kinetics of phase transitions. Namely, our analysis addresses the following system of two parabolic equations in the unknowns ϑ and χ

$$\varepsilon\vartheta_t + \lambda\chi_t - \Delta\left(-\frac{1}{\vartheta}\right) = f \quad \text{in } \Omega \times (0, T) \tag{1}$$

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$$\delta\chi_t - \Delta\chi + \beta(\chi) + \sigma'(\chi) + \frac{\lambda}{\vartheta} - \frac{\lambda}{\vartheta_c} \ni 0 \quad \text{in } \Omega\times(0,T) \tag{2}$$

where ε, δ are positive coefficients, Ω is a bounded, connected domain in $\mathbb{R}^N$, $N=1,2,3$, with smooth boundary $\partial\Omega$, occupied by a physical system subject to a phase transition in the time interval $(0,T)$. The evolution of the system is described in terms of its absolute temperature ϑ (whereas ϑ_c is the critical temperature of the phase change), and of the order parameter χ, which is characteristic of the specific phase transition considered and may represent, e.g. the fraction of lattice sites with pointing up spins in the mean field theory for Ising ferromagnets, or the local proportion of the solid or liquid phase in a liquid–solid phase transition. In this framework, the constant $\lambda>0$ is the latent heat density, f possibly represents a heat source, the maximal monotone operator $\beta:\mathbb{R}\to 2^{\mathbb{R}}$ is the subdifferential of a convex function $\hat{\beta}$, σ' is a Lipschitz continuous function and the term $\beta+\sigma'$ is the derivative of a non-convex free energy potential $\mathscr{W}$; for instance, $\mathscr{W}$ may have the double-well potential form

$$\mathscr{W}(x) = \frac{(x^2-1)^2}{4}, \quad x\in\mathbb{R} \tag{3}$$

The differential inclusion (2) yields an Allen–Cahn type dynamics for the order parameter χ, while (1) renders the balance law

$$\partial_t e + \operatorname{div}\mathbf{q} = f$$

for the internal energy of the system $e := \vartheta + \lambda\chi$, when the heat flux $\mathbf{q}$ is postulated to have the form

$$\mathbf{q} = -\nabla(\ell(\vartheta)), \quad \text{for } \ell(\vartheta) = -\frac{1}{\vartheta} \tag{4}$$

Note that the coefficient ε of ϑ_t could be interpreted as the specific heat, the term $\varepsilon\vartheta_t$ deriving from the *purely caloric* part of the total free energy functional associated to the system—see also Reference [3]—while δ is the coefficient of the interfacial energy term. Further, as it is clear from the *variational* derivation developed in Reference [1] (see also Reference [4]), the system (1)–(2) is *thermodynamically consistent*, namely in agreement with the second law of thermodynamics.

Let us now slightly modify (2) by incorporating the term $-\lambda/\vartheta_c$ in the Lipschitz function σ' and normalizing the constant λ to 1. If we now set $\varepsilon=0$ in the first equation (1) and formally plug (2) into (1), the Penrose–Fife system then reduces to

$$\chi_t - \Delta(\delta\chi_t - \Delta\chi + \beta(\chi) + \sigma'(\chi)) \ni f \quad \text{in } \Omega\times(0,T) \tag{5}$$

Accordingly, letting $\varepsilon=\delta=0$ in (1)–(2) and proceeding in the same way, we formally obtain

$$\chi_t - \Delta(-\Delta\chi + \beta(\chi) + \sigma'(\chi)) \ni f \quad \text{in } \Omega\times(0,T) \tag{6}$$

In fact, when $\mathfrak{W}$ has the double well form (3), and thus

$$\beta(\chi) + \sigma'(\chi) = \chi^3 - \chi \tag{7}$$

(6) reduces indeed to the standard Cahn–Hilliard equation, which models the phenomenon of phase separation in a binary mixture rapidly cooled down from an initially spatially homogeneous state into a miscibility gap. In this framework, the order parameter χ represents the

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local concentration of one of the two components. The general form (6) for the Cahn–Hilliard equation was proposed in Reference [5] (and further investigated in Reference [6]), possibly to account for, e.g. a constraint on the values of the order parameter χ, such as $0\leqslant\chi\leqslant 1$, which could be incorporated in the maximal monotone operator β. Note that, when $f\equiv 0$ and (6) is supplemented with the initial condition

$$\chi(\cdot,0)=\chi_0 \tag{8}$$

and the boundary conditions

$$\partial_n\chi=\partial_n(-\Delta\chi+\xi+\sigma'(\chi))=0 \quad \text{in } \partial\Omega\times(0,T) \tag{9}$$

(where ∂_n denotes the outward normal derivative on $\partial\Omega$ and ξ is a pointwise selection in $\beta(x)$ fulfilling (6) with the equality), then we can infer from (6) that

$$\frac{1}{|\Omega|}\int_\Omega \chi(x,t)\,\mathrm{d}x=\frac{1}{|\Omega|}\int_\Omega \chi_0(x)\,\mathrm{d}x \quad \text{for a.e. } t\in(0,T)$$

i.e. χ is a *conserved parameter*, in agreement with its physical interpretation as concentration.

On the other hand, (5) was first introduced (in the form corresponding to (7) and with $f\equiv 0$) in Reference [7] to model viscosity effects in the phase separation of, e.g. polymer–polymer systems. Subsequently, Gurtin [8] independently obtained (5) (with the choice (7) for the non-linearity $\beta+\sigma'$), as a particular case of a generalized Cahn–Hilliard equation he had introduced taking into account the working of internal microforces in the model for phase separation. We refer to Reference [9] for a detailed survey of the well-posedness results so far proved in this connection. As for the Cahn–Hilliard equation, one can introduce a non-linearity given by a maximal monotone graph in the standard viscous Cahn–Hilliard equation and thus get (5). Once again, when $f\equiv 0$ and (5) is coupled with (8) and the homogeneous Neumann boundary conditions (9), χ is a conserved parameter.

The formal arguments we have developed to obtain (5) and (6) suggest the possibility of investigating whether the viscous Cahn–Hilliard and the Cahn–Hilliard equations can be actually viewed as singular limits of the system (1)–(2) for vanishing ε (which corresponds to the case in which the specific heat tends to zero), and vanishing ε and δ. Analogous asymptotic analyses were carried out for the phase field model proposed by Caginalp [10] (which can be interpreted as a *linearization* of the Penrose–Fife phase field system, see Reference [4]), in a series of papers [11–13]; we also mention [14,15], in which the asymptotic behaviour of the attractors associated to the Caginalp phase field system was examined for the same vanishing time relaxation parameters. As for the (non-conserved) Penrose–Fife phase field model, we may also quote the two papers [16,17]. There, an asymptotic analysis (for δ and/or the coefficient of $-\Delta\chi$ in (2) vanishing), is worked out in the particular case $\beta=H^{-1}$, H^{-1} being the inverse of the Heaviside graph.

In our framework, the question thus arises of checking if the solutions of the initial-boundary value problem $\mathbf{P}_\varepsilon^*$ given by (1)–(2), supplemented with

$$\partial_n\chi=\partial_n\left(-\frac{1}{\vartheta}\right)=0 \quad \text{in } \partial\Omega\times(0,T) \tag{10}$$

and the initial conditions

$$\chi(\cdot,0)=\chi_0, \quad \vartheta(\cdot,0)=\vartheta_0 \tag{11}$$

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do converge to a solution of the problem ((5), (8), (9)) (((6), (8), (9)), respectively), when $\varepsilon \downarrow 0$ (when $\varepsilon, \delta \downarrow 0$, respectively).

In fact, as we shall see later on (cf. Theorem 3 and Remark 2.3), the system (1)–(2) turns out to be a bad approximation both for (5) and for (6). To fix ideas, let us focus on the case of vanishing ε and fixed $\delta > 0$—analogous considerations hold for ε, $\delta \downarrow 0$—and let us conveniently rephrase the *candidate* limiting equation (5) by means of an auxiliary variable u, which in the literature is known as *chemical potential*:

$$\chi_t - \Delta u = f \quad \text{a.e. in } Q \tag{12}$$

$$\delta \chi_t - \Delta \chi + \xi + \sigma'(\chi) = u, \quad \xi \in \beta(\chi), \quad \text{a.e. in } Q \tag{13}$$

where we have set $Q := \Omega \times (0, T)$. Then, let $\{(\vartheta_*^\varepsilon, \chi_*^\varepsilon)\}$ be a sequence of solutions to $\mathbf{P}_\varepsilon^*$, converging to a solution χ of the viscous Cahn–Hilliard equation as $\varepsilon \downarrow 0$: in view of the structure (12)–(13) of the limiting equation, we may expect the sequence $-1/\vartheta_*^\varepsilon$ to converge to u in a suitable topology, which might possibly be strong enough to entail

$$u = \lim_{\varepsilon \downarrow 0} -\frac{1}{\vartheta_*^\varepsilon} \leqslant 0 \quad \text{a.e. in } Q \tag{14}$$

as pointed out in Remark 2.3 later on. However, a sign constraint on the chemical potential does not *pertain* to the viscous Cahn–Hilliard equation; in turn, it is substantially due to the heat flux law (4) postulated for the balance equation (1). Further, the choice (4) does not even ensure sufficient *a priori* estimates on $\{\vartheta_*^\varepsilon\}$ in order to pass to the limit in (1): as a matter of fact, no estimate is available on (a positive power of) $\{\vartheta_*^\varepsilon\}$. These technical drawbacks are latent in the fact that we can prove only a *partial* convergence result for the solutions to $\mathbf{P}_\varepsilon^*$ when $\varepsilon \downarrow 0$, see Theorem 3 in the next section.

We are thus naturally led to approximate (5) and (6) by a phase field system featuring an alternative form for the heat flux law. Actually, the previous arguments somehow suggest to replace the map ℓ in (4) with a strictly increasing function, which beaves like $-1/x$ only for x positive and $x \ll 1$, and which allows instead for a linear growth for $x \gg 1$. The corresponding heat flux law was first introduced in its full generality—featuring growths intermediate between the logarithmic and the linear ones—in References [18,19], based on the consideration that the form (4) is well-suited to model low and intermediate temperatures, but not large ones. Therefore, we will approximate (5) and (6) with the system

$$\varepsilon \vartheta_t + \chi_t - \Delta \left(\varepsilon^\eta \vartheta - \frac{1}{\vartheta} \right) = f \quad \text{in } \Omega \times (0, T) \tag{15}$$

$$\delta \chi_t - \Delta \chi + \xi + \sigma'(\chi) = \varepsilon^\eta \vartheta - \frac{1}{\vartheta}, \quad \xi \in \beta(\chi), \quad \text{in } \Omega \times (0, T) \tag{16}$$

where η is a fixed parameter, $0 < \eta < 1$. Let us emphasize that we have replaced the occurrence $\ell(\vartheta) = -1/\vartheta$ in *both* (1) and (2) with the term $\alpha_\varepsilon(\vartheta)$, where

$$\alpha_\varepsilon(r) := \varepsilon^\eta r - \frac{1}{r}, \quad r \in D(\alpha_\varepsilon) = (0, +\infty) \tag{17}$$

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In fact, unlike the *standard* phase field system (1)–(2), in the approximating system (15)–(16) we have introduced, the coupling term $\alpha_\varepsilon(\vartheta)$ in the second equation has a linear growth at infinity.

Henceforth, we will refer to the initial-boundary value problem given by (15)–(16), supplemented with (11) and homogeneous Neumann boundary conditions for χ and $\alpha_\varepsilon(\vartheta)$ (in agreement with the previous (10)), as Problem $\mathbf{P}_\varepsilon$.

In general, the problem arises of establishing a well-posedness result for the phase field system

$$k_1\vartheta_t + \lambda\chi_t - \Delta u = f, \quad u \in \alpha(\vartheta), \quad \text{in } \Omega \times (0,T) \tag{18}$$

$$k_2\chi_t - \Delta\chi + \xi + \sigma'(\chi) = \lambda u, \quad \xi \in \beta(\chi), \quad \text{in } \Omega \times (0,T) \tag{19}$$

where α is in general a *maximal monotone graph* on $\mathbb{R}^2$ and k_1, k_2, λ positive constants. We will couple (18)–(19) with (11) and the boundary conditions

$$\partial_n\chi = \partial_n u = 0 \quad \text{in } \partial\Omega \times (0,T) \tag{20}$$

The main analytical difficulties connected with this initial-boundary value problem are due to the presence of a double non-linearity (given by the two maximal monotone operators α and β) and to the boundary conditions (20). In fact, third type boundary conditions are generally prescribed for ϑ (see References [16,20,21]), or for u. Indeed, the latter sort of boundary conditions on u

$$\partial_n u + \gamma u = h \quad \text{on } \partial\Omega \times (0,T)$$

would make the problem easier to handle, by allowing to recover a $H^1(\Omega)$-bound for u from the first equation (18) by means of Poincaré's inequality. In this connection, let us mention the well-posedness results of References [22] and [3] for the phase field system (1)–(2) with this kind of boundary conditions. In particular, in the latter paper the analysis is focused on the graph $\alpha(r) = -1/r$, $r>0$, but as we will see later on, it could be easily adapted to a general maximal monotone operator α, provided that boundary conditions of the third type on u are considered. On the other hand, a well-posedness result with homogeneous Neumann boundary conditions for u (in the case of the standard heat flux law (4)) has been obtained in Reference [23], but in one space dimension only, whereas in Reference [24] non-homogeneous Neumann boundary conditions are considered in three space dimensions, but under an additional 'compatibility' condition on the data of the problem. Finally, let us mention the recent work [25], in which an existence and regularity result is demonstrated for the Penrose–Fife system with the special heat flux law of Reference [18] and with non-homogeneous Neumann boundary conditions. In this paper, the lack of 'coercivity' due to the choice of the boundary condition for u is substantially overcome by introducing an additional growth hypothesis on the convex potential $\hat{\beta}$ of the graph β.

We will adopt the same hypothesis (see (28) in the next section) for handling the system ((11), (18), (19), (20)). Further, we will adapt the subdifferential approach devised in Reference [21] in order to tackle the double non-linearity given by the two maximal monotone operators α and β, and we will thus prove our main existence result Theorem 1, which ensures in particular the well-posedness of $\mathbf{P}_\varepsilon$, with the choices $k_1 = \varepsilon$, $k_2 = \delta$ and $\alpha = \alpha_\varepsilon$ in the system

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(18)–(19). We will show—see Theorem 2—that the sequence of solutions to $\mathbf{P}_\varepsilon$ thus obtained converges as $\varepsilon\downarrow 0$ (and for fixed $\delta>0$) to the unique solution of the initial-boundary value problem ((8), (9), (12), (13)), providing as well some error estimates specifying the rate of the convergences proved. As a byproduct, we will also obtain a well-posedness result for the viscous Cahn–Hilliard equation in the general form (5), which has not yet been proved, to our knowledge. Subsequently, we will develop an analogous asymptotic analysis, with error estimates, for the approximate system given by (15) coupled to

$$\delta\chi_t - \Delta\chi + \chi^3 - \chi = \varepsilon^n\vartheta - \frac{1}{\vartheta}, \quad \text{in } \Omega\times(0,T) \tag{21}$$

for vanishing ε and δ. Hereafter, we will refer to the problem (15), (21), with the initial and boundary conditions (11) and (20) for χ and $\varepsilon^n\vartheta - 1/\vartheta$, as Problem $\mathbf{P}^*_{\varepsilon\delta}$. We will obtain in the limit the unique solution to the standard Cahn–Hilliard equation with source term, see Theorem 4.

Plan of the paper: The main existence and convergence results are stated in Section 2, where the notation and some preliminary material on Mosco convergence are recalled as well. Section 3 is devoted to the proof of our well-posedness result Theorem 1: the argument relies on an approximation procedure and on a further regularization, which are developed in detail throughout the Sections 3.1, 3.2, and it features the tools of variational convergence introduced in the previous section. The asymptotic analyses of $\mathbf{P}_\varepsilon$ for vanishing ε (and fixed δ) and of $\mathbf{P}^*_{\varepsilon\delta}$ for ε, $\delta\downarrow 0$ are finally developed in Sections 4 and 5, respectively. In the latter section Theorem 3 is proved as well.

2. MAIN RESULTS

2.1. Notation

We set

$$H := L^2(\Omega), \quad V := H^1(\Omega), \quad \text{and } W := \{v\in H^2(\Omega) : \partial_n v = 0\}$$

and identify H with its dual space H', recalling that the embeddings

$$W \subset V \subset H \subset V' \subset W'$$

are dense and compact. We will denote by $(\cdot,\cdot)_H$ the inner product in H, by $\langle\cdot,\cdot\rangle$, the duality pairing between V' and V, and by $|\cdot|_H$, $\|\cdot\|_V$, $\|\cdot\|_{V'}$ the norms in H, in V and in V'. Let $\mathfrak{W}$, $\mathscr{V}$, $\mathscr{H}$, and $\mathscr{V}'$ be the subspaces of the elements v with zero mean value $m(v) = (1/|\Omega|)\langle v, 1\rangle$ in W, V, H, and V'. Let us also introduce the operator $A : V\to V'$ given by

$$\langle Au, v\rangle := \int_\Omega \nabla u\nabla v\,\mathrm{d}x \quad \forall u, v\in V$$

A is the realization of the Laplace operator with homogeneous Neumann boundary conditions. Indeed, we note that $Au\in\mathscr{V}'$ for every $u\in V$ and that $Au = -\Delta u\in H$ whenever $u\in W$. Moreover, the restriction of A to $\mathscr{V}$ is an isomorphism and we can thus consider the inverse operator $\mathscr{N} : \mathscr{V}'\to\mathscr{V}$, defined by

$$A(\mathscr{N}v) = v \quad \forall v\in\mathscr{V}'$$

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Clearly, $\mathcal{N}v \in \mathcal{W}$ if $v \in \mathcal{H}$. Throughout the remainder of this paper, we will make a systematic use of the relations

$$\langle Au, \mathcal{N}v\rangle = \langle v, u\rangle \quad \forall u \in V, \ \forall v \in \mathcal{V}' \tag{22}$$

$$\langle u, \mathcal{N}v\rangle = \int_\Omega \nabla(\mathcal{N}u)\nabla(\mathcal{N}v)\,dx = \langle v, \mathcal{N}u\rangle \quad \forall u, v \in \mathcal{V}' \tag{23}$$

which entail that the following norms on V and V'

$$\|u\|_V^2 := \langle Au, u\rangle + (u, m(u))_H \quad \forall u \in V \tag{24}$$

$$\|v\|_{V'}^2 := \langle v, \mathcal{N}(v - m(v))\rangle + (v, m(v))_H \quad \forall u \in V' \tag{25}$$

are equivalent to the standard ones on behalf of Poincaré's inequality for the zero mean value functions.

2.2. *Statement of the main results*

We here enlist our main assumptions on the system (18)–(19):

$$\alpha \text{ is a maximal monotone graph on } \mathbb{R} \text{ with domain } D(\alpha), \text{ and} \tag{26}$$
$$\alpha = \partial\hat{\alpha}, \text{ for a proper, convex function } \hat{\alpha}: D(\hat{\alpha}) \to [0, +\infty),$$

$$\beta: \mathbb{R} \to 2^{\mathbb{R}} \text{ is a maximal monotone graph,} \tag{27}$$
$$\beta = \partial\hat{\beta}, \text{ with } \hat{\beta}: \mathbb{R} \to [0, +\infty) \text{ a convex function s.t. } \hat{\beta}(0) = 0 \text{ and}$$

$$\exists M_\beta \geqslant 0 \quad \|\xi| \leqslant M_\beta(1 + \hat{\beta}(r)) \quad \forall \xi \in \beta(r), \quad \forall r \in \mathbb{R} \tag{28}$$

$$\sigma \in C^1(\mathbb{R}) \text{ and } \sigma': \mathbb{R} \to \mathbb{R} \text{ is Lipschitz continuous with Lipschitz constant } L. \tag{29}$$

Note that (28) allows β to have at most an exponential growth, see also Reference [25].

In this framework, we can give a *variational* formulation of the initial-boundary value problem for (18)–(19) corresponding to the data

$$\chi_0 \in V, \quad \hat{\beta}(\chi_0) \in L^1(\Omega) \tag{30}$$

$$\vartheta_0 \in H, \quad \vartheta_0 \in D(\alpha) \text{ a.e. in } \Omega, \quad \hat{\alpha}(\vartheta_0) \in L^1(\Omega) \tag{31}$$

$$f \in L^2(0, T; H) \tag{32}$$

Problem 1

Find $\vartheta \in H^1(0, T; V') \cap L^\infty([0, T]; H)(\subset C^0_w([0, T]; H))$ and $\chi \in L^\infty(0, T; V) \cap H^1(0, T; H) \cap L^2(0, T; W)(\subset C^0_w([0, T]; V))$ such that $\vartheta \in D(\alpha)$, $\chi \in D(\beta)$ a.e. in Q,

$$k_1\partial_t\vartheta + \lambda\partial_t\chi + Au = f \quad \text{in } V', \text{ a.e. in } (0, T) \tag{33}$$

$$\text{for some } u \in L^2(0,T;V) \text{ with } u \in \alpha(\vartheta) \text{ a.e. in } Q$$

$$k_2\partial_t\chi + A\chi + \xi + \sigma'(\chi) = \lambda u \quad \text{in } H, \text{ a.e. in } (0,T), \tag{34}$$

$$\text{for some } \xi \in L^2(0,T;H) \text{ with } \xi \in \beta(\chi) \text{ a.e in } Q$$

and subject to the initial conditions (11).

Note that the boundary conditions (20) for u and χ are implied by the variational formulation (33)–(34); $C^0_w([0,T];H)$ and $C^0_w([0,T];V)$ denote the spaces of functions weakly continuous on $[0,T]$ with values in H and V, respectively.

The following continuous dependence result holds, ensuring that if Problem 1 admits a solution, it is necessarily *unique*.

Proposition 2.1
Let $(\vartheta^1_0, \chi^1_0, f^1)$ and $(\vartheta^2_0, \chi^2_0, f^2)$ be two triplets of data for Problem 1 fulfilling (30)–(32) and let (ϑ^i, χ^i), $i = 1,2$, be the corresponding solutions. Set

$$M^* := \max_{i=1,2}\{\|\hat{\alpha}(\vartheta^i_0)\|_{L^1(\Omega)} + \|\chi^i_0\|^2_V + \|\hat{\beta}(\chi^i_0)\|_{L^1(\Omega)} + \|f^i\|^2_{L^2(0,T;V')}\}$$

Then there exists a positive constant C_1, depending on $M^*, T, |\Omega|, k_1, k_2, \lambda$, and L (the Lipschitz constant of σ'), such that

$$\|\vartheta^1 - \vartheta^2\|_{C^0([0,T];V')} + \|\chi^1 - \chi^2\|_{C^0([0,T];H)\cap L^2(0,T;V)}$$
$$\leqslant C_1(\|\vartheta^1_0 - \vartheta^2_0\|_{V'} + |\chi^1_0 - \chi^2_0|_H + \|f^1 - f^2\|_{L^2(0,T;V')}) \tag{35}$$

Theorem 1
Assume that (26)–(29) and (30)–(32) hold. Then Problem 1 admits a unique solution (ϑ, χ).

The viscous Cahn–Hilliard equation as singular limit of the Penrose–Fife phase field: We can now rigorously formulate the initial-boundary value problem for the viscous Cahn–Hilliard equation (12)–(13) (setting $\delta = 1$), with non-linearities.

Problem 2
Let $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ and σ fulfil (27)–(29). Given χ_0 and f such that

$$\chi_0 \in V, \quad \hat{\beta}(\chi_0) \in L^1(\Omega) \tag{36}$$

$$f \in L^2(0,T;V'), \quad m(f(t)) = 0 \quad \text{for a.e. } t \in (0,T) \tag{37}$$

find $\chi \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)$ and $u \in L^2(0,T;V)$ satisfying

$$\partial_t\chi + Au = f \quad \text{in } V', \text{ a.e. in } (0,T) \tag{38}$$

$$\partial_t\chi + A\chi + \xi + \sigma'(\chi) = u \quad \text{in } H, \text{ a.e. in } (0,T) \tag{39}$$

$$\text{for some } \xi \in L^2(0,T;H) \text{ with } \xi \in \beta(\chi) \text{ a.e. in } Q$$

and such that the initial condition (8) holds for χ.

A continuous dependence statement can be obtained for Problem 2, as well.

Proposition 2.2
Let (χ_{01}, f_1) and (χ_{02}, f_2) be two pairs of data for Problem 2 fulfilling (36) and (37), and let $\chi_i,\ i=1,2$, be the corresponding solutions. Set

$$M_* := \max_{i=1,2}\{\|\chi_{0i}\|_V^2 + \|\hat{\beta}(\chi_{0i})\|_{L^1(\Omega)} + \|f_i\|^2_{L^2(0,T;V')}\}$$

Then there exists a positive constant C_2, depending on M_*, T, $|\Omega|$ and L, such that

$$\|\chi_1 - \chi_2\|_{C^0([0,T];H)\cap L^2(0,T;V)} \leqslant C_2(|\chi_{01} - \chi_{02}|_H + \|f_1 - f_2\|_{L^2(0,T;V')}) \tag{40}$$

To prove the existence of solutions to Problem 2, we proceed by approximation. First of all, given the data (χ_0, f) as in (36)–(37), we construct two approximating sequences $\{\chi_\varepsilon^0\}$ and $\{f^\varepsilon\}$ such that

$$\chi_\varepsilon^0 \in V, \quad \hat{\beta}(\chi_\varepsilon^0) \in L^1(\Omega)\ \forall \varepsilon > 0, \quad \text{and}\ \chi_\varepsilon^0 \to \chi_0\ \text{in}\ H \tag{41}$$

$$f^\varepsilon \in L^2(0,T;H) \quad \forall \varepsilon > 0,\ f^\varepsilon \to f\ \text{in}\ L^2(0,T;V') \tag{42}$$

We can now consider for any $\varepsilon > 0$ the quadruple $\{(\chi_\varepsilon, \vartheta_\varepsilon, u_\varepsilon, \xi_\varepsilon)\}$ (where $u_\varepsilon = \alpha_\varepsilon(\vartheta_\varepsilon) = \varepsilon^\eta \vartheta_\varepsilon - 1/\vartheta_\varepsilon$ and ξ_ε is a selection in $\beta(\chi_\varepsilon)$), which fulfils the variational formulation (33)–(34) of Problem $\mathbf{P}_\varepsilon$ ((11), (15), (16), (20))—note that, since we are interested in the case of *vanishing* ε only, we will systematically set $\delta = 1$ in (16)—supplemented with the source term f^ε and with the initial data $(\chi_\varepsilon^0, \vartheta_\varepsilon^0)$, $\{\vartheta_\varepsilon^0\}$ being a sequence with the property

$$\vartheta_\varepsilon^0 \in H, \quad \vartheta_\varepsilon^0(x) > 0 \quad \text{for a.e. } x \in \Omega, \quad \hat{\alpha}_\varepsilon(\vartheta_\varepsilon^0) \in L^1(\Omega) \quad \forall \varepsilon > 0 \tag{43}$$

The following convergence result states that, under suitable assumptions on the sequences of initial data $\{\chi_\varepsilon^0\}$ and $\{\vartheta_\varepsilon^0\}$, the sequence $\{\chi_\varepsilon\}$ thus constructed converges as $\varepsilon \downarrow 0$ to the (unique) solution χ of Problem 2.

Theorem 2

(i) Let (χ_0, f) comply with (36)–(37), assume that the sequences of data $\{\chi_\varepsilon^0\}$, $\{\vartheta_\varepsilon^0\}$, and $\{f^\varepsilon\}$ satisfy (41)–(43), and that there exists a positive constant K such that

$$\varepsilon^{(1+\eta)/2}|\vartheta_\varepsilon^0|_H + \varepsilon \int_\Omega (\log(\vartheta_\varepsilon^0(x)))^- \,\mathrm{d}x \leqslant K \quad \forall \varepsilon > 0 \tag{44}$$

$$\|\chi_\varepsilon^0\|_V + \|\hat{\beta}(\chi_\varepsilon^0)\|_{L^1(\Omega)} \leqslant K \quad \forall \varepsilon > 0 \tag{45}$$

where $(\log(\vartheta_\varepsilon^0(x)))^- = \max(-\log(\vartheta_\varepsilon^0(x)), 0)$. Let $(\chi_\varepsilon, \vartheta_\varepsilon)$ be the solution to $\mathbf{P}_\varepsilon$: then there exists a triplet (χ, u, ξ) such that the following convergences hold for $\{\chi_\varepsilon\}$ and $\{\vartheta_\varepsilon\}$ as $\varepsilon \downarrow 0$, and for $\{u_\varepsilon\}$, $\{\xi_\varepsilon\}$ along a subsequence $\{\varepsilon_k\}$

$$\chi_\varepsilon \rightharpoonup^* \chi \quad \text{in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W) \tag{46}$$

$$\chi_\varepsilon \to \chi \quad \text{in } C^0([0,T];H) \cap L^2(0,T;V) \tag{47}$$

$$\varepsilon\vartheta_\varepsilon \to 0 \quad \text{in } L^\infty(0,T;H),\ \text{and}\ \varepsilon\vartheta_\varepsilon \rightharpoonup 0 \quad \text{in } H^1(0,T;V') \tag{48}$$

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$$u_{\varepsilon_k} \rightharpoonup u \quad \text{as } k\uparrow\infty \text{ in } L^2(0,T;V) \tag{49}$$

$$\xi_{\varepsilon k} \rightharpoonup \xi \quad \text{as } k\uparrow\infty \text{ in } L^2(0,T;H), \text{ and } \xi\in\beta(\chi) \text{ a.e. in } Q \tag{50}$$

Moreover, (χ, u, ξ) fulfil (38)–(39) and the initial condition (8), namely χ is the unique solution to Problem 2.

(ii) There exists a constant $C_3 \geqslant 0$, depending on T, $|\Omega|$ and L only, such that the error estimates

$$\|\chi_\varepsilon - \chi\|_{C^0([0,T];H)\cap L^2(0,T;V)}$$
$$\leqslant C_3(\varepsilon^{(1-\eta)/4} + \|\chi_\varepsilon^0 - \chi_0\|_{V'} + |\chi_\varepsilon^0 - \chi_0|_H + \|f - f^\varepsilon\|_{L^2(0,T;V')}) \tag{51}$$

$$\|\varepsilon\vartheta_\varepsilon\|_{L^\infty(0,T;H)} \leqslant C\varepsilon^{(1-\eta)/2} \tag{52}$$

hold for every $\varepsilon\in(0,1)$.

It is clear from (51) that the *smaller* η is, the *better* is the error estimate for $\chi_\varepsilon - \chi$, and subsequently the more accurate is the approximation of the viscous Cahn–Hilliard equation provided by the system (15)–(16).

Finally, let us perform a asymptotic analysis of the standard Penrose–Fife phase-field system (1)–(2) for vanishing ε and fixed δ (say $\delta=1$). To this aim, we approximate the data χ_0 and f of Problem 2 by two sequences $\{\chi_{*\varepsilon}^0\}$ and $\{f_*^\varepsilon\}$ fulfilling (41)–(42), and we also consider a sequence $\{\vartheta_{*\varepsilon}^0\}$ satisfying (43), as well as

$$\varepsilon\|\vartheta_{*\varepsilon}^0\|_{V'} \to 0 \quad \text{as } \varepsilon\downarrow 0 \tag{53}$$

For any $\varepsilon>0$, we consider the quadruple $(\chi_*^\varepsilon, \vartheta_*^\varepsilon, u_*^\varepsilon, \xi_*^\varepsilon)$ satisfying

$$\varepsilon\partial_t\vartheta_*^\varepsilon + \partial_t\chi_*^\varepsilon + Au_*^\varepsilon = f_*^\varepsilon, \quad u_*^\varepsilon = -\frac{1}{\vartheta_*^\varepsilon}, \quad \text{in } V', \text{ a.e. in } (0,T) \tag{54}$$

$$\partial_t\chi_*^\varepsilon + A\chi_*^\varepsilon + \xi_*^\varepsilon + \sigma'(\chi_*^\varepsilon) = u_*^\varepsilon, \quad \xi_*^\varepsilon\in\beta(\chi_*^\varepsilon), \quad \text{in } H, \text{ a.e. in } (0,T) \tag{55}$$

namely the variational formulation of Problem $\mathbf{P}_\varepsilon^*$ (with $\lambda=\delta=1$), supplemented with the data $\vartheta_{*\varepsilon}^0$, $\chi_{*\varepsilon}^0$, and f_*^ε.

Theorem 3

Let χ be the unique solution to Problem 2 supplemented with the data χ_0 and f, and let us consider a pair (u,ξ) satisfying (38)–(39). Suppose that u fulfils the additional hypothesis

$$u(x,t)\leqslant 0 \quad \text{for a.e. } (x,t)\in Q, \quad \text{and} \quad \exists\vartheta\in L^2(Q) \quad \text{s.t. } u=\alpha(\vartheta) \text{ a.e. in } Q \tag{56}$$

(i) Then the following convergences hold for the sequence $\{(\chi_*^\varepsilon, \vartheta_*^\varepsilon, u_*^\varepsilon, \xi_*^\varepsilon)\}$ as $\varepsilon\downarrow 0$:

$$\chi_*^\varepsilon \to \chi \quad \text{in } C^0([0,T];H)\cap L^2(0,T;V)$$

$$\chi_*^\varepsilon \rightharpoonup^* \chi \quad \text{in } H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W) \tag{57}$$

Besides, for any subsequence $\{\varepsilon_k\}$ such that $\{u^{\varepsilon_k}\}$ weakly converges as $k \uparrow \infty$ to some function u^* in $L^2(0,T;V)$, there holds

$$u^{\varepsilon_k} \rightharpoonup u^* \text{ in } L^2(0,T;V) \Longrightarrow \nabla u^* = \nabla u \quad \text{a.e. in } Q \tag{58}$$

Remark 2.3
This result is far from being satisfying: in fact, the sequence $\{\chi_*^\varepsilon\}$ is shown to converge to the solution of the viscous Cahn–Hilliard equation only provided that we know *a priori* that such a solution exists—which is by the way ensured by Theorem 2. What is more, the argument works under the additional assumption (56) on the *chemical potential u*, which is not natural in the framework of the Cahn–Hilliard equation.

On the other hand, as we have already mentioned in the Introduction and will be clear from the proof of Theorem 3, if the systems (1)–(2) and (12)–(13) are to be compared, in view of the a priori estimates on $\{u_*^\varepsilon\}$ we may expect $u_*^\varepsilon \rightharpoonup u$ in $L^2(0,T;V)$, up to a subsequence. Therefore, the sign constraint (14) on the limiting chemical potential ensues, and the first part of (56) turns out to be a necessary condition for (57) to hold.

The Cahn–Hilliard equation as singular limit of the Penrose–Fife phase-field: Let us first of all recall the variational formulation (already given in Reference [13]), for the limiting initial boundary value problem associated to the standard Cahn–Hilliard equation (6)–(7) with source term f, and supplemented with the initial and boundary conditions (8) and (9). As for Problem 2, our formulation features an auxiliary variable u.

Problem 3
Given $\chi_0 \in V$ and f as in (37), find $\chi \in H^1(0,T;V') \cap L^\infty(0,T;V) \cap L^2(0,T;W)$ and $u \in L^2(0,T;V)$ such that

$$\partial_t \chi + Au = f \quad \text{in } V', \text{ a.e. in } (0,T) \tag{59}$$

$$A\chi + \chi^3 - \chi = u \quad \text{in } H, \text{ a.e. in } (0,T) \tag{60}$$

and the initial condition (8) holds.

An existence theorem and a continuous dependence result, entailing uniqueness, for Problem 3 have already been established in Reference [13, Theorems 2.4, 2.5]. We are now going to investigate the approximation of Problem 3 by means of the solutions to Problem $\mathbf{P}_{\varepsilon\delta}^*$ ((11), (15), (20), (21)) for vanishing ε *and* δ. As a first step, we approximate the data χ^0 and f of Problem 3: namely, we consider three sequences $\{\chi_{\varepsilon\delta}^0\}$, $\{\vartheta_{\varepsilon\delta}^0\}$ and $\{f_{\varepsilon\delta}\}$ such that

$$\chi_{\varepsilon\delta}^0 \in V \quad \forall \varepsilon, \delta > 0, \quad \chi_{\varepsilon\delta}^0 \to \chi_0 \quad \text{in } H \text{ as } \varepsilon, \delta \downarrow 0 \tag{61}$$

$$\vartheta_{\varepsilon\delta}^0 \in H, \vartheta_{\varepsilon\delta}^0(x) > 0 \quad \text{for a.e. } x \in \Omega, \ \log(\vartheta_{\varepsilon\delta}^0) \in L^1(\Omega), \quad \forall \varepsilon, \delta > 0 \tag{62}$$

$$f_{\varepsilon\delta} \in L^2(0,T;H) \quad \forall \varepsilon, \delta > 0, \quad f_{\varepsilon\delta} \to f \quad \text{in } L^2(0,T;V') \quad \text{as } \varepsilon, \delta \downarrow 0 \tag{63}$$

Theorem 4

(i) Given $\chi_0 \in V$ and f fulfilling (37), let $\{(\chi_{\varepsilon\delta}, \vartheta_{\varepsilon\delta})\}$ be the sequence of pairs solving Problem $\mathbf{P}_{\varepsilon\delta}^*$ for every ε, $\delta > 0$, supplemented with the approximate initial data $\{\chi_{\varepsilon\delta}^0\}$, $\{\vartheta_{\varepsilon\delta}^0\}$ fulfilling (61)–(62) and the source terms $\{f_{\varepsilon\delta}\}$ as in (63); let us denote by $\{u_{\varepsilon\delta}\}$

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the sequence $\{\varepsilon^{\eta}\vartheta_{\varepsilon\delta} - 1/\vartheta_{\varepsilon\delta}\}$. Suppose moreover that (44) and (45) hold for $\{\vartheta^0_{\varepsilon\delta}\}$ and $\{\chi^0_{\varepsilon\delta}\}$.

Then there exists a (unique) pair (χ, u) such that the following convergences hold for the sequence $\{(\chi_{\varepsilon\delta}, \vartheta_{\varepsilon\delta})\}$ as $\varepsilon,\ \delta\downarrow 0$:

$$\chi_{\varepsilon\delta} \rightharpoonup^* \chi \quad \text{in } L^\infty(0,T;V)\cap L^2(0,T;W) \tag{64}$$

$$\chi_{\varepsilon\delta} \to \chi \quad \text{in } C^0([0,T];V')\cap L^2(0,T;V) \tag{65}$$

$$u_{\varepsilon\delta} \rightharpoonup u \quad \text{in } L^2(0,T;V) \tag{66}$$

$$\varepsilon\vartheta_{\varepsilon\delta} \to 0 \quad \text{in } L^\infty(0,T;H) \tag{67}$$

$$\delta\partial_t\chi_{\varepsilon\delta} \to 0 \quad \text{in } L^2(0,T;H) \tag{68}$$

$$\varepsilon\vartheta_{\varepsilon\delta} + \chi_{\varepsilon\delta} \rightharpoonup \chi \quad \text{in } H^1(0,T;V') \tag{69}$$

Furthermore, $\chi\in C^0([0,T];H)$ and it is the unique solution to Problem 3.

(ii) There exists a constant $C_4\geqslant 0$, only depending on T and $|\Omega|$, such that the error estimates (52) for $\|\varepsilon\vartheta_{\varepsilon\delta}\|_{L^\infty(0,T;H)}$ and

$$\|\chi - \chi_{\varepsilon\delta}\|_{C^0([0,T];V')\cap L^2(0,T;V)}$$
$$\leqslant C_4(\|\chi_0 - \chi^0_{\varepsilon\delta}\|_{V'} + \delta^{1/2}|\chi_0 - \chi^0_{\varepsilon\delta}|_H + \|f - f_{\varepsilon\delta}\|_{L^2(0,T;V')} + \varepsilon^{(1-\eta)/4} + \delta) \tag{70}$$

hold for every $\varepsilon,\ \delta\in(0,1)$.

2.3. *Preliminary results*

2.3.1. G-convergence. Let $\mathscr{H}$ be a Hilbert space with scalar product $(\cdot,\cdot)_{\mathscr{H}}$: we say that a sequence $\mathscr{T}_n:\mathscr{H}\to 2^{\mathscr{H}}$ of maximal monotone operators converges to a maximal monotone operator $\mathscr{T}$ on $\mathscr{H}$ *in the sense of G-convergence* (or *in the sense of graphs*), if $\forall[x,y]\in\mathscr{T}$ there exists a sequence $[x_n,y_n]\in\mathscr{T}_n$ such that $[x_n,y_n]\to[x,y]$ strongly in $\mathscr{H}\times\mathscr{H}$.

A crucial property of G-convergence (which can be retrieved in the proof of [26, Proposition 3.59, p. 361]) we will exploit later on is that, when $\mathscr{T}_n$ G-converges to $\mathscr{T}$, then

$$\left\{[x_n,y_n]\in\mathscr{T}_n, x_n\rightharpoonup x,\ y_n\rightharpoonup y \text{ in } \mathscr{H} \text{ and } \liminf_{n\uparrow\infty}(x_n,y_n)_{\mathscr{H}}\leqslant(x,y)_{\mathscr{H}}\right\}$$
$$\Rightarrow [x,y]\in\mathscr{T} \tag{71}$$

Let us now recall the definition of *Mosco convergence*, which will play a basic role in the proof of our existence result Theorem 1 in the following section.

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2.3.2. Mosco convergence. Let $\{\psi_n\}, \psi : \mathscr{H} \to \mathbb{R} \cup \{+\infty\}$ be proper, convex l.s.c. functionals on $\mathscr{H}$: we say that $\{\psi_n\}$ converges to ψ *in the sense of Mosco* if

- $\forall z \in \mathscr{H}$ there exists a sequence $z_n \to z$ such that $\psi_n(z_n) \to \psi(z)$
- $\forall z \in \mathscr{H}$ and $\forall z_n \rightharpoonup z$, $\psi(z) \leqslant \liminf_{n\uparrow\infty} \psi_n(z_n)$.

The link between the two convergence notions we have introduced is yielded by [26, Theorem 3.66]:

$$\text{if } \psi_n \to \psi \text{ in the sense of Mosco, then } \partial\psi_n \ G\text{-converges to } \partial\psi \tag{72}$$

$\partial\psi_n$ ($\partial\psi$, resp.) denoting the subdifferential (in the sense of convex analysis) of ψ_n (ψ, resp.).

Throughout the next sections, we will also make use of the two following inequalities for the functional $\mathscr{W}'$ (the derivative of the double well potential $\mathscr{W}$ in (3)):

$$\forall \rho \in [0,1) \quad \exists C_p \geqslant 0 : \quad \mathscr{W}'(r)r \geqslant \rho r^4 - C_p \ \forall r \in \mathbb{R} \tag{73}$$

$$\forall \mu > 0 \quad \exists C_\mu > 0 \quad \mathscr{W}'(r) \leqslant \mu r^4 + C_\mu \ \forall r \in \mathbb{R} \tag{74}$$

Throughout the proofs of Theorems 1–4, we will adopt the general convention of denoting by the same symbol C several constants depending only on the quantities specified in the statement of each theorem, and possibly on the initial data; we will specifically point out the few occurring exceptions.

3. WELL-POSEDNESS FOR PROBLEM 1

Throughout this section, we will prove a well-posedness result for the system (18)–(19) in which the constants k_1, k_2 and λ are normalized to 1.

3.1. An approximate problem

Let v be a positive constant; following Reference [24], we introduce the system consisting of (19) and (75), where

$$\vartheta_t + \chi_t + vu - \Delta u = f, \quad u \in \alpha(\vartheta), \quad \text{in } \Omega \times (0,T) \tag{75}$$

with homogeneous Neumann boundary conditions on u and on χ. Note that the additional term vu in (75) makes the approximate problem 'coercive', in the sense that the operator $J^v : V \to V'$ defined by

$$\langle J^v v, w \rangle := v \int_\Omega vw \, \mathrm{d}x + \int_\Omega \nabla v \nabla w \, \mathrm{d}x \quad \forall v, w \in V \tag{76}$$

is linear, continuous and coercive. Throughout this one and the following subsection 3.2, we will endow the space V with the inner product $((\cdot,\cdot))$ given by $((v_1, v_2)) := \langle J^v v_1, v_2 \rangle = \langle J^v v_2, v_1 \rangle \forall v_1, v_2 \in V$, and, accordingly, the dual space V' with the inner product $((w_1, w_2))_* := \langle w_1, (J^v)^{-1} w_2 \rangle = \langle w_2, (J^v)^{-1} w_1 \rangle \forall w_1, w_2 \in V', J^v$ being therefore the duality mapping between V and V'.

We can now give a variational formulation of the initial-boundary value problem associated to (75) and (19).

Problem $\mathbf{P}^{\nu}$
Given the data χ_0, ϑ_0 and f fulfilling (30)–(32), and α, β, σ as in (26)–(29), find $\vartheta \in H^1(0,T;V') \cap L^\infty(0,T;H)$ and $\chi \in L^\infty(0,T;V) \cap H^1(0,T;H) \cap L^2(0,T;W)$ such that $\vartheta \in D(\alpha)$, $\chi \in D(\beta)$ a.e. in Q,

$$\begin{aligned}&\partial_t\vartheta + \partial_t\chi + J^{\nu}u = f \ \text{ in } V', \ \text{ a.e. in } (0,T),\\ &\text{for some } u \in L^2(0,T;V) \text{ with } u \in \alpha(\vartheta) \text{ a.e. in } Q\end{aligned} \tag{77}$$

$$\begin{aligned}&\partial_t\chi + A\chi + \xi + \sigma'(\chi) = u \ \text{ in } H, \ \text{ a.e. in } (0,T),\\ &\text{for some } \xi \in L^2(0,T;H) \text{ with } \xi \in \beta(\chi) \text{ a.e. in } Q\end{aligned} \tag{78}$$

and the pair (χ,ϑ) is subject to the initial conditions (11).

As far as the existence issue of Problem $\mathbf{P}^{\nu}$ is concerned, we will adopt a subdifferential operator approach to deal with the double nonlinearity of the system (77)–(78). Namely, like in References [3,27] (see also Reference [28], where the same subdifferential technique is developed for a Volterra integrodifferential equation), we will interpret (77) as an evolution equation in the space V' generated by a subdifferential operator associated to the graph α, and we will accordingly provide a generalized formulation of Problem $\mathbf{P}^{\nu}$.

Indeed, let us define the functional $\phi: H \to \mathbb{R} \cup \{+\infty\}$ by

$$\phi(v) := \begin{cases} \displaystyle\int_\Omega \hat{\alpha}(v(x))\mathrm{d}x & \text{if } \hat{\alpha}(v) \in L^1(\Omega) \\ +\infty & \text{otherwise} \end{cases}$$

Note that ϕ is convex and l.s.c. on H, but its extension $\bar{\phi}$ by $+\infty$ on V' is not l.s.c. on V', in general. Therefore, we can consider the l.s.c. envelope of $\bar{\phi}$ on V', namely the functional $\varphi: V' \to \mathbb{R} \cup \{+\infty\}$ given by

$$\varphi(v) := \inf\left\{ \liminf_{n\uparrow+\infty} \bar{\phi}(v_n) : v_n \text{ in } H, \quad v_n \to v \text{ in } V' \right\}, \quad v \in V'$$

By definition, φ is proper, l.s.c. and convex on V'; it has been proved (see Reference [27, Cor. 1.6]) that φ coincides with ϕ on H and that the subdifferential $\partial_{V'}\varphi: V' \to 2^{V'}$ of φ in the Hilbert space V' fulfils

$$\partial_{V'}\varphi(v) = \{J^{\nu}w \in V' : w \in V, \quad w \in \alpha(v) \text{ a.e. in } \Omega\}, \quad \forall v \in D(\partial_{V'}\varphi) \cap H \tag{79}$$

Hence, we are naturally led to introduce the following extension of Problem $\mathbf{P}^{\nu}$: note that less regularity is required on the initial data and the source term.

Problem $\mathbf{P}^{\nu}_*$
Given

$$\chi_0^{\nu} \in V, \ \hat{\beta}(\chi_0^{\nu}) \in L^1(\Omega), \quad \vartheta_0^{\nu} \in V' \cap D(\varphi), \quad f^{\nu} \in L^2(0,T;V') \tag{80}$$

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find $\vartheta \in H^1(0,T;V')$ and $\chi \in L^\infty(0,T;V) \cap H^1(0,T;H) \cap L^2(0,T;W)$ such that $\vartheta(t) \in D(\partial_{V'}\varphi)$ for a.e. $t \in (0,T)$,

$$\partial_t \vartheta + \partial_t \chi + J^\nu u = f^\nu \text{ in } V', \text{ a.e. in } (0,T),$$

$$\text{for some } u \in L^2(0,T;V) \text{ such that } J^\nu u(t) \in \partial_{V'}\varphi(\vartheta(t)) \text{ for a.e. } t \in (0,T) \tag{81}$$

and (78) holds, with the initial conditions

$$\vartheta(\cdot,0) = \vartheta_0^\nu, \quad \chi(\cdot,0) = \chi_0^\nu \tag{82}$$

When $(\chi_0^\nu, \vartheta_0^\nu, f^\nu) = (\chi_0, \vartheta_0, f)$, the unique solution (ϑ,χ) of Problem $\mathbf{P}^\nu$ clearly solves Problem $\mathbf{P}_*^\nu$. On the other hand, on behalf of (79) we realize that any solution (ϑ,χ) of $\mathbf{P}_*^\nu$ is a solution of $\mathbf{P}^\nu$ as well, provided that $\vartheta \in C_w^0([0,T];H)$.

Proposition 3.1
Assume χ_0^ν, ϑ_0^ν and f^ν fulfil (80): then Problem $\mathbf{P}_*^\nu$ admits a unique solution (ϑ^ν,χ^ν); further, under the additional regularity assumptions

$$f^\nu \in L^2(0,T;H), \quad \vartheta_0^\nu \in D(\varphi) \cap L^2(\Omega) \tag{83}$$

we have that $\vartheta^\nu \in C_w^0([0,T];H)$ and (ϑ^ν,χ^ν) is a solution of Problem $\mathbf{P}^\nu$ as well.

3.2. Well-posedness for the approximate problem

The uniqueness part of Proposition 3.1—which entails the uniqueness of the solutions to Problem $\mathbf{P}^\nu$ as well—can be shown like the uniqueness results of References [24, Prop. 6.2] and [3, Theorem. 3.1]. As for the existence, we will just outline the main steps of the proof and leave the technicalities aside.

As in Reference [27, Sect. 3], we introduce a further regularization of Problem $\mathbf{P}_*^\nu$: namely, under the additional hypothesis that $\alpha(0) \ni 0$ (which we can assume without loss of generality just to perform this regularization procedure, as it is pointed out in Reference [27]), we consider the sequence $\{\alpha_{\lambda_n}\}$ of the Yosida regularizations of the maximal monotone operator α (see e.g. [29]), for a sequence $\lambda_n \downarrow 0$ as $n \uparrow \infty$, and we accordingly define the functions $\alpha_n : \mathbb{R} \to \mathbb{R}$

$$\alpha_n(r) := \alpha_{\lambda_n}(r) + \frac{r}{n}, \quad \text{with primitives } \widehat{\alpha_n}(r) := \widehat{\alpha_{\lambda_n}}(r) + \frac{r^2}{2n}, \quad r \in \mathbb{R} \tag{84}$$

$\{\widehat{\alpha_{\lambda_n}}\}$ being the sequence of the Yosida approximations of $\hat{\alpha}$. Note that α_n is a Lipschitz continuous increasing function for every $n \in \mathbb{N}$; finally, let us introduce the functionals $\varphi_n : V' \to \mathbb{R} \cup \{+\infty\}$

$$\varphi_n(v) := \begin{cases} \int_\Omega \widehat{\alpha_n}(v(x))\,\mathrm{d}x & \text{if } v \in H \\ +\infty & \text{if } v \in V' \backslash H \end{cases}$$

The φ_n are coercive in H, as it can be readily verified from (84), so that it is easy to see that, besides being convex, they are also l.s.c. on V'. Let us point out that (see Reference

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[27] for the proofs)

$$\partial_{V'}\varphi_n(v)=\{J^\nu w\in V': w\in V,\ w\in\alpha_n(v) \text{ a.e. in } \Omega\},\ v\in D(\partial_{V'}\varphi_n)\cap H \tag{85}$$

$$\varphi_n\to\varphi \text{ as } n\uparrow\infty \text{ in the sense of Mosco in } V' \tag{86}$$

hence by (72) $\partial_{V'}\varphi_n\to\partial_{V'}\varphi_n$ in the sense of graphs.

Let us now consider the multivalued operators $\mathscr{A}_n: L^2(0,T;V')\to 2^{L^2(0,T;V')}$ given by

$$v\in\mathscr{A}_n(u) \text{ iff } v(t)\in\partial_{V'}\varphi_n(u(t)) \text{ for a.e. } t\in(0,T)$$

for every $u\in L^2(0,T;V')$, and accordingly let $\mathscr{A}$ be the realization of $\partial_{V'\varphi}$ on $L^2(0,T;V')$.

The operators $\mathscr{A}_n,\mathscr{A}$ are maximal monotone—see [29, Ex. 2.1.3, 2.3.3]. Furthermore, using e.g. References [26, Prop. 3.60] and [29, Prop. 2.16], it is easy to check that the graph convergence of $\partial_{V'}\varphi_n$ to $\partial_{V'}\varphi$ in $V'\times V'$ induces the convergence

$$\mathscr{A}_n\ G\text{-converges to } \mathscr{A} \text{ in } L^2(0,T;V')\times L^2(0,T;V' \text{ as } n\uparrow\infty \tag{87}$$

Finally, we introduce the sequence $\{\beta_n\}$ of the Yosida regularizations of the graph β, with $\beta_n=\partial\widehat{\beta_n}$; note that

$$\widehat{\beta_n}(r)=\min_{s\in\mathbb{R}}\left(\frac{|r-s|^2}{2\lambda_n}+\hat{\beta}(s)\right)\geqslant 0\ \forall r\in\mathbb{R}$$

since $\hat{\beta}$ takes positive values by (27).

We can approximate the data χ_0^ν, ϑ_0^ν and f^ν of Problem $\mathbf{P}_*^\nu$: namely, there exist (see Reference [3, Lemma 4.2]) three sequences $\{\vartheta_{0n}\}$, $\{\chi_{0n}\}$ and $\{f_n\}$ such that

$$\vartheta_{0n}\in V \text{ for every } n\in\mathbb{N}, \vartheta_{0n}\to\vartheta_0^\nu \text{ in } V', \text{ and } \varphi_n(\vartheta_{0n})\to\varphi(\vartheta_0^\nu) \text{ as } n\uparrow\infty \tag{88}$$

$$\begin{gathered}\chi_{0n}\in W \text{ for every } n\in\mathbb{N}, \chi_{0n}\to\chi_0^\nu \text{ in } V, \text{ and}\\ \int_\Omega(\hat{\beta}_n(\chi_{0n})+\sigma(\chi_{0n}))\,\mathrm{d}x\to\int_\Omega(\hat{\beta}(\chi_0^\nu)+\sigma(\chi_0^\nu)\,\mathrm{d}x) \text{ as } n\uparrow\infty\end{gathered} \tag{89}$$

$$f_n\in H^1(0,T;H)\ \forall n \text{ and } f_n\to f^\nu \text{ in } L^2(0,T;V') \text{ as } n\uparrow\infty \tag{90}$$

We can now introduce for every $n\in\mathbb{N}$ the following initial-boundary value problem.

Problem $\mathbf{P}_n^\nu$

Given ϑ_{0n}, χ_{0n} and f_n fulfilling (88)–(90), find $\vartheta_n^\nu\in H^1(0,T;H)\cap L^\infty(0,T;V)$ with $J^\nu\alpha_n(\vartheta_n^\nu)\in L^2(0,T;H)$, and $\chi_n^\nu\in L^\infty(0,T;W)\cap H^1(0,T;V)$ satisfying

$$\partial_t\vartheta_n^\nu+\partial_t\chi_n^\nu+J^\nu\alpha_n(\vartheta_n^\nu)=f_n \text{ in } H, \text{ a.e. in } (0,T) \tag{91}$$

$$\partial_t\chi_n^\nu+A\chi_n^\nu+\beta_n(\chi_n^\nu)+\sigma'(\chi_n^\nu)=\alpha_n(\vartheta_n^\nu) \text{ in } H, \text{ a.e. in } (0,T) \tag{92}$$

$$\chi_n^\nu(\cdot,0)=\chi_{0n},\quad \vartheta_n^\nu(\cdot,0)=\vartheta_{0n} \tag{93}$$

The proof of an existence and uniqueness result for Problem $\mathbf{P}_n^\nu$ can be easily deduced from the results in Reference [30]. We will now show that the solutions to this regularized problem

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converge to the unique solution of Problem $\mathbf{P}^{\nu}_{*}$, and, under the further assumption (83), of $\mathbf{P}^{\nu}$.

Proposition 3.2
Let $\{(\vartheta^{\nu}_{n}, \chi^{\nu}_{n})\}$ be the sequence of solutions to Problem $\mathbf{P}^{\nu}_{n}$: there exists a quadruple of functions $\chi^{\nu} \in H^1(0,T;H) \cap L^{\infty}(0,T;V) \cap L^2(0,T;W)$, $\vartheta^{\nu} \in H^1(0,T;V')$, $u^{\nu} \in L^2(0,T;V)$ and $\xi^{\nu} \in L^2(0,T;H)$ such that $\vartheta^{\nu}(t) \in D(\partial_{V'}\varphi)$ and $J^{\nu}u^{\nu}(t) \in \partial_{V'}\varphi(\vartheta^{\nu}(t))$ for a.e. $t \in (0,T)$, $\chi^{\nu} \in D(\beta)$ and $\xi^{\nu} \in \beta(\chi^{\nu})$ a.e. in Q, and the following convergences hold as $n \uparrow \infty$

$$\chi^{\nu}_{n} \rightharpoonup^{*} \chi^{\nu} \text{ in } H^1(0,T;H) \cap L^{\infty}(0,T;V) \cap L^2(0,T;W) \tag{94}$$

$$\chi^{\nu}_{n} \to \chi^{\nu} \text{ in } C^0([0,T];H) \cap L^2(0,T;V) \tag{95}$$

$$\vartheta^{\nu}_{n} \rightharpoonup \vartheta^{\nu} \text{ in } H^1(0,T;V') \tag{96}$$

$$\alpha_n(\vartheta^{\nu}_{n}) \rightharpoonup u^{\nu} \text{ in } L^2(0,T;V) \tag{97}$$

$$\beta_n(\chi^{\nu}_{n}) \rightharpoonup \xi^{\nu} \text{ in } L^2(0,T;H) \tag{98}$$

Moreover, $(\vartheta^{\nu}, \chi^{\nu}, u^{\nu}, \zeta^{\nu})$ fulfil (77), (78) and (82), so that $(\vartheta^{\nu}, \chi^{\nu})$ is the unique solution to Problem $\mathbf{P}^{\nu}_{*}$. Further, if (83) holds and $\vartheta_{0n} \rightharpoonup \vartheta^{\nu}_{0}$ in H, there holds

$$\begin{aligned} &\vartheta^{\nu}_{n} \rightharpoonup^{*} \vartheta^{\nu} \text{ in } L^{\infty}(0,T;H),\ \vartheta^{\nu} \in C^0_w([0,T];H) \text{ and} \\ &\vartheta^{\nu}_{n} \to \vartheta^{\nu} \text{ in } C^0([0,T];V'). \end{aligned} \tag{99}$$

Obviously, proving Proposition 3.1 reduces now to demonstrating the statement above.

Proof
In order to prove (94)–(98), we first obtain some *a priori estimates* on the sequence of solutions $\{\vartheta^{\nu}_{n}, \chi^{\nu}_{n}\}$ in suitable norms. First, we test (91) by $\alpha_n(\vartheta^{\nu}_{n})$, (92) by $\partial_t \chi^{\nu}_{n}$, add the resulting equations and integrate on $(0,t)$, $t \in (0,T)$. Two terms cancel out and we thus obtain

$$\begin{aligned} &\int_{\Omega} \hat{\alpha}_n(\vartheta^{\nu}_{n}(x,t))\,\mathrm{d}x + \nu \int_0^t |\alpha_n(\vartheta^{\nu}_{n}(s))|^2_H\,\mathrm{d}s + \int_0^t |\nabla \alpha_n(\vartheta^{\nu}_{n}(s))|^2_H\,\mathrm{d}s + \int_0^t |\partial_t \chi^{\nu}_{n}(s)|^2_H\,\mathrm{d}s \\ &\quad + \frac{1}{2}|\nabla \chi^{\nu}_{n}(t)|^2_H + \int_{\Omega} \hat{\beta}_n(\chi^{\nu}_{n}(x,t))\,\mathrm{d}x + \int_0^t \int_{\Omega} \sigma'(\chi^{\nu}_{n}(x,s))\partial_t \chi^{\nu}_{n}(x,s)\,\mathrm{d}x\,\mathrm{d}s \\ &= \int_{\Omega} \hat{\alpha}_n(\vartheta_{0n}(x))\,\mathrm{d}x + \frac{1}{2}|\nabla \chi_{0n}|^2_H + \int_{\Omega} \hat{\beta}_n(\chi_{0n}(x))\,\mathrm{d}x + \int_0^t \langle f_n(s), \alpha_n(\vartheta^{\nu}_{n}(s))\rangle\,\mathrm{d}s \end{aligned} \tag{100}$$

Let us remark that we have the full V-norm of $\alpha_n(\vartheta^{\nu}_{n})$ on the left-hand side of (100), so that we estimate the last summand on the right-hand side of the above relation by

$$\int_0^t \langle f_n(s), \alpha_n(\vartheta^{\nu}_{n}(s))\rangle\,\mathrm{d}s \leqslant \frac{1}{2}\int_0^t \|f_n(s)\|^2_{V'}\,\mathrm{d}s + \frac{1}{2}\int_0^t \|\alpha_n(\vartheta^{\nu}_{n}(s))\|^2_V\,\mathrm{d}s \tag{101}$$

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Moreover, on behalf of the Lipschitz continuity of σ', we deduce for the last term on the left-hand side of (100)

$$\left|\int_0^t (\sigma'(\chi_n^v(s)), \partial_t \chi_n^v(s))_H \, ds\right| \leqslant \frac{1}{2}\int_0^t |\partial_t \chi_n^v(s)|_H^2 \, ds + \int_0^t |\sigma'(\chi_{0n})|_H^2 \, ds$$

$$+ L^2 \int_0^t |\chi_n^v(s) - \chi_{0n}|_H^2 \, ds \qquad (102)$$

(where $L \geqslant 0$ is the Lipschitz constant of σ'), and we can estimate the latter term by

$$L^2 \int_0^t |\chi_n^v(s) - \chi_{0n}|_H^2 \, ds \leqslant L^2 T \int_0^t \|\partial_t \chi_n^v\|_{L^2(0,s;H)}^2 \, ds \qquad (103)$$

Collecting (100)–(103), taking into account the assumptions (88)–(90) on the data ϑ_{0n}, χ_{0n} and f_n, and finally applying Gronwall's Lemma (see, e.g. Reference [29, Lemma A.4]) to $\|\partial_t \chi_n^v\|_{L^2(0,t;H)}^2$, we easily conclude that

$$\|\chi_n^v\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} + \|\alpha_n(\vartheta_n^v)\|_{L^2(0,T;V)} + \|\hat{\alpha}_n(\vartheta_n^v)\|_{L^\infty(0,T;L^1(\Omega))}$$

$$+ \|\hat{\beta}_n(\chi_n^v)\|_{L^\infty(0,T;L^1(\Omega))} \leqslant C, \qquad (104)$$

for a constant $C \geqslant 0$ *independent* of n, (but possibly depending on v). By comparison in (91), we also infer that

$$\|\partial_t \vartheta_n^v\|_{L^2(0,T;V')} \leqslant C \qquad (105)$$

It is straightforward to see that

$$\|\sigma'(\chi_n^v)\|_{L^2(0,T;H)} \leqslant C$$

since $\sigma' \in C^{0,1}(\mathbb{R})$; then, testing (92) by $A\chi_n^v$ and using that the function β_n is monotone increasing, we obtain that $\{A\chi_n^v\}$ is bounded in $L^2(0,T;H)$, so that by standard elliptic regularity results we have that

$$\|\chi_n^v\|_{L^2(0,T;W)} \leqslant C, \quad \text{as well as } \|\beta_n(\chi_n^v)\|_{L^2(0,T;H)} \leqslant C, \qquad (106)$$

the last estimate following from a comparison argument in the second equation.

The previous estimates (104)–(106) and well known weak compactness results [31, Theorem 4, Cor. 5] ensure that there exist $u^v \in L^2(0,T;V)$, $\chi^v \in L^\infty(0,T;V) \cap H^1(0,T;H) \cap L^2(0,T;W)$, $\vartheta^v \in H^1(0,T;V')$ and $\xi^v \in L^2(0,T;H)$ such that the convergences (94)–(98) hold along some subsequences (which we will not specify for simplicity of notation) of $\{\alpha_n(\vartheta_n^v)\}$, $\{\chi_n^v\}$, $\{\vartheta_n^v\}$ and $\{\beta_n(\chi_n^v)\}$; note that (88) and (96) ensure that $\vartheta_n^v \to \vartheta^v$ in $C_w^0([0,T];V')$. Now, β obviously induces a maximal monotone graph on $L^2(Q)$: so, in order to check that $\xi^v \in \beta(\chi^v)$, by Barbu [32, Proposition 1.1] it suffices to remark that

$$\limsup_{n\uparrow\infty} \iint_Q \beta^n(\chi_n^v)\chi_n^v \, dt \, dx \leqslant \iint_Q \xi^v \chi^v \, dt \, dx \qquad (107)$$

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To see this, we test (92) by χ_n^v and integrate on $(0, T)$, noting that the left-hand side of (107) equals

$$\limsup_{n\uparrow\infty} \int\int_Q (-\partial_t(\chi_n^v)\chi_n^v - |\nabla\chi_n^v|^2 - \sigma'(\chi_n^v)\chi_n^v + \chi_n^v\alpha_n(\vartheta_n^v))\,dt\,dx$$

$$= \limsup_{n\uparrow\infty}(|\chi_{0n}|_H^2 - |\chi_n^v(T)|_H^2 - \|\nabla\chi_n^v\|_{L^2(0,T;H)}^2 + \int_0^T ((\alpha_n(\vartheta_n^v), \chi_n^v)_H - (\sigma'(\chi_n^v), \chi_n^v)_H)\,dt)$$

$$= |\chi_0^v|_H^2 - |\chi^v(T)|_H^2 - \|\nabla\chi^v\|_{L^2(0,T;H)}^2 + \int_0^T ((u^v, \chi^v)_H - (\sigma'(\chi^v), \chi^v)_H)\,dt$$

where the second equality can be obtained in view of on the assumption (89) on the approximating initial data and on the convergences (95) for $\{\chi_n^v\}$—which guarantee that $\sigma'(\chi_n^v) \to \sigma'(\chi^v)$ strongly in $L^2(0, T; H)$—and (97). Note that the last member in the above chain of inequalities equals the right-hand side of (107), as it is readily checked by passing to the limit in (92) in view of (94)–(98).

Finally it remains to check that ϑ^v, χ^v and u^v fulfil the first equation: noting (90), (94), (96) and (97), which entails that

$$J^v(\alpha_n(\vartheta_n^v)) \rightharpoonup J^v u^v \quad \text{in } L^2(0, T; V') \tag{108}$$

it all reduces to proving that

$$J^v u^v(t) \in \partial_{V'}\varphi(\vartheta^v(t)) \quad \text{for a.e } t \in (0, T)$$

which can be rephrased by means of the multivalued operator $\mathscr{A}$ previously introduced as

$$J^v u^v \in \mathscr{A} \tag{109}$$

Then, taking into account (71), (87), (96), and (108), we can conclude (109) provided that

$$\limsup_{n\uparrow\infty} \int_0^T ((J^v(\alpha_n(\vartheta_n^v(t))), \vartheta_n^v(t)))_*\,dt \leqslant \int_0^T ((J^v(u^v(t)), \vartheta^v(t)))_*\,dt$$

Thus, recalling the definition of $((\cdot,\cdot))_*$, we test (91) by $(J^v)^{-1}(\vartheta^v)$. The convergences so far obtained allow us to pass to the lim sup in (91), obtaining the above inequality (we hereby refer to the proof of Reference [3, Theorem 4.1], where all the computations are developed in detail).

We have thus proved that the limiting pair (ϑ^v, χ^v) is the (unique) solution to Problem $\mathbf{P}_*^v$: since the limits do not depend on the subsequences extracted, we finally infer that the convergences (94)–(96) (and thus (97), (98) hold indeed for the *whole* sequences $\{\chi_n^v\}$, $\{\vartheta_n^v\}$, $\{\alpha_n(\vartheta_n^v)\}$ and $\{\beta^n(\chi_n^v)\}$.

In the end, it remains to see the additional regularity $\vartheta^v \in C_w^0(0, T; H)$. Indeed, upon assuming (83) and strengthening (90) up to $f_n \to f^v$ in $L^2(0, T; H)$, we can test (91) by $\vartheta_n^v(s)$

(which is in V for a.e. $s \in (0,T)$), then integrate on $(0,t)$ and get

$$|\vartheta_n^\nu(t)|_H^2 + \int_0^t (\nabla\alpha_n(\vartheta_n^\nu(s)), \nabla\vartheta_n^\nu(s))_H \, ds \leqslant |\vartheta_{0n}|_H^2 + \frac{3}{2}\|\vartheta_n^\nu\|_{L^2(0,t;H)}^2$$

$$+ \frac{1}{2}(\|f^n\|_{L^2(0,T;H)}^2 + \|\partial_t\chi_n^\nu\|_{L^2(0,T;H)}^2 + \nu^2\|\alpha_n(\vartheta_n^\nu)\|_{L^2(0,t;H)}^2)$$

By the monotonicity of α_n, the integral term on the left-hand side is positive, and applying Gronwall's Lemma, we easily infer

$$\|\vartheta_n^\nu\|_{L^\infty(0,T;H)} \leqslant C \tag{110}$$

which eventually entails (99) by virtue of [31, Cor. 5]. □

3.3. Passage to the limit and existence for Problem 1

The following convergence result holds for the sequence $\{(\chi^\nu, \vartheta^\nu, u^\nu, \xi^\nu)\}_\nu$ of the solutions to Problem $\mathbf{P}^\nu$.

Proposition 3.3
Let $\{(\chi^\nu, \vartheta^\nu, u^\nu, \xi^\nu)\}_\nu$ be the sequence of solutions to Problem $\mathbf{P}^\nu$: then there exist $\chi \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \vartheta \in H^1(0,T;V') \cap L^\infty(0,T;H), u \in L^2(0,T;V)$ and $\xi \in L^2(0,T;H)$ such that the following convergences hold as $\nu \downarrow 0$ and along some subsequence $\{\nu_k\}$ as $k \uparrow \infty$

$$\chi^\nu \rightharpoonup^* \chi \text{ in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W) \tag{111}$$

$$\chi^\nu \to \chi \text{ in } C^0([0,T]);H) \cap L^2(0,T;V) \tag{112}$$

$$\vartheta^\nu \rightharpoonup^* \vartheta \text{ in } H^1(0,T;V') \cap L^\infty(0,T;H) \tag{113}$$

$$\vartheta^\nu \rightharpoonup \vartheta \text{ in } C^0([0,T]);V') \tag{114}$$

$$u^{\nu k} \rightharpoonup u \text{ in } L^2(0,T;V), \nu u^\nu \to 0 \text{ in } L^2(0,T;H) \tag{115}$$

$$\xi^{\nu k} \rightharpoonup \xi \text{ in } L^2(0,T;H) \tag{116}$$

and the quadruple $(\chi, \vartheta, u, \xi)$ satisfies (33)–(34) and (11).

Proof
In order to provide some *a priori* estimates for the approximate solutions $\{\vartheta^\nu\}$, $\{\chi^\nu\}$, let us test (77) by u^ν, (78) by $\partial_t\chi^\nu$ and integrate on $(0,t)$ for a.e. $t \in (0,T)$. We retrieve the analogue—obviously replacing ϑ_n^ν by ϑ^ν, χ_n^ν by χ^ν and $\alpha_n(\vartheta_n^\nu)$ by u^ν—of (100) in the proof of Proposition 3.2. (Note by the way that we need to perform a regularization procedure on the test function $\partial_t\chi^\nu(s)$, which does not belong to V for a.e. $s \in (0,T)$: to this purpose, we refer the reader to [33, Lemma 2.1], where the suitable regularization is devised, and to [13, Proof of Theorem 3], where the technique is applied in detail to a similar estimate). However, unlike (100), we do not have the full V-norm of u^ν on the left-hand side, since ν

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is a *vanishing* parameter now. So we remark that

$$\left|\int_0^t \langle f(s), u^\nu(s)\rangle \, \mathrm{d}s\right| \leqslant \frac{1}{2}\int_0^t \|f(s)\|_{V'}^2 \, \mathrm{d}s + \frac{1}{2}\int_0^t \|u^\nu(s) - m(u^\nu(s))\|_V^2 \, \mathrm{d}s$$

$$+ \int_0^t \|f(s)\|_{V'} \|m(u^\nu(s))\|_V \, \mathrm{d}s \tag{117}$$

By comparison in (78),

$$\int_0^t \|f(s)\|_{V'} \|m(u^\nu(s))\|_V \, \mathrm{d}s$$

$$\leqslant |\Omega|^{1/2} \int_0^t \|f(s)\|_{V'} (|m(\partial_t \chi^\nu(s))| + |m(\xi^\nu(s))| + |m(\sigma'(\chi^\nu(s)))|) \, \mathrm{d}s \tag{118}$$

Labelling $S_i, i = 1,2,3$ the summands on the right-hand side of (118), using (28) and the Lipschitz continuity σ', we can see that

$$S_1 \leqslant c_1 \|f\|_{L^2(0,T;V')}^2 + \frac{1}{4}\|\partial_t \chi^\nu\|_{L^2(0,t;H)}^2 \tag{119}$$

$$S_2 \leqslant c_2 \int_0^t \|f(s)\|_{V'} \left(\int_\Omega (1 + \hat{\beta}(\chi^\nu(x,s))) \, \mathrm{d}x\right) \mathrm{d}s \tag{120}$$

$$S_3 \leqslant \frac{1}{2}\|f\|_{L^2(0,T;V')}^2 + c_3 \left(1 + \int_0^t \|\partial_t \chi^\nu\|_{L^2(0,s;H)}^2 \, \mathrm{d}s\right) \tag{121}$$

where the constants c_1, c_2, and c_3 depend on $|\Omega|$, T, the initial datum χ_0, the Lipschitz constant L of σ' and M_β in (28).

Collecting (117)–(121), the analogue of the estimate (100) now reads

$$\int_\Omega \hat{\alpha}(\vartheta^\nu(x,t)) \, \mathrm{d}x + \nu \int_0^t |u^\nu(s))|_H^2 \, \mathrm{d}s + \int_0^t |\nabla u^\nu(s))|_H^2 \, \mathrm{d}s$$

$$+ \int_0^t |\partial_t \chi^\nu(s)|_H^2 \, \mathrm{d}s + \frac{1}{2}|\nabla \chi^\nu(t)|_H^2 + \int_\Omega \hat{\beta}(\chi^\nu(x,t)) \, \mathrm{d}x \leqslant C_0(1 + \|f\|_{L^2(0,T;V')}^2)$$

$$+ \left|\int_0^t (\sigma'(\chi^\nu(s)), \partial_t \chi^\nu(s))_H \, \mathrm{d}s\right| + \frac{1}{2}\|u^\nu - m(u^\nu)\|_{L^2(0,t;V)}^2 + \frac{1}{4}\|\partial_t \chi^\nu\|_{L^2(0,t;H)}^2$$

$$+ C\left(\int_0^t \left[\|f(s)\|_{V'} \left(\int_\Omega (1 + \hat{\beta}(\chi^\nu(x,s))) \, \mathrm{d}x\right) + \|\partial_t \chi^\nu\|_{L^2(0,s;H)}^2\right] \mathrm{d}s\right) \tag{122}$$

the constant C_0 depending on the initial data ϑ_0, χ_0. Now we can deal with the second term on the right-hand side of (122) arguing as in (102)–(103), while, on behalf of (24), we have

$$\frac{1}{2}\|u^\nu - m(u^\nu)\|^2_{L^2(0,t;V)} = \frac{1}{2}\int_0^t |\nabla u^\nu(s))|^2_H \, \mathrm{d}s \tag{123}$$

and the latter term above is controlled by the third integral on the left-hand side of (122). Finally, in order to estimate the last summand on the right-hand side, we apply Gronwall's Lemma [29, Lemma A.4] to $\|\hat{\beta}(\chi^\nu(s))\|_{L^1(\Omega)} + \|\partial_t\chi^\nu\|^2_{L^2(0,s;H)}$ and conclude that

$$\|\chi^\nu\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} + \nu^{1/2}\|u^\nu\|_{L^2(0,T;H)} + \|u^\nu - m(u^\nu)\|_{L^2(0,T;V)} + \|\hat{\alpha}(\vartheta^\nu)\|_{L^\infty(0,T;L^1(\Omega))} + \|\hat{\beta}(\chi^\nu)\|_{L^\infty(0,T;L^1(\Omega))} \leqslant C \tag{124}$$

for a constant C independent of ν.

Recalling (28) and arguing by comparison in (78), we note that $\|m(u^\nu)\|_{L^2(0,T)} \leqslant C$, so that we also infer that

$$\|u^\nu\|_{L^2(0,T;V)} \leqslant C \tag{125}$$

Moreover, since the estimates (105), (106), and (110) in the proof of Proposition 3.2 are in fact *independent* of ν, we deduce that

$$\|\vartheta^\nu\|_{H^1(0,T;V')} + \|\vartheta^\nu\|_{L^\infty(0,T;H)} + \|\chi^\nu\|_{L^2(0,T;W)} + \|\xi^\nu\|_{L^2(0,T;H)} \leqslant C$$

So there exist χ, ϑ, u, and ξ such that the convergences (111)–(116) hold along some subsequence $\{\nu_k\}$.

We pass to the limit in (77)–(78) and, arguing in the same way as to prove (107), we note that

$$\limsup_{\nu\downarrow 0} \int\int_Q \chi^\nu\xi^\nu \, \mathrm{d}x\,\mathrm{d}t \leqslant \int\int_Q \chi\xi \, \mathrm{d}x\,\mathrm{d}t \tag{126}$$

entailing that $\xi \in \beta(\chi)$ a.e. in Q in view of [32, Prop 1.1]. In the same way, $u \in \alpha(\vartheta)$ a.e. in Q follows from

$$\limsup_{\nu\downarrow 0} \int\int_Q \vartheta^\nu u^\nu \, \mathrm{d}x\,\mathrm{d}t \leqslant \int\int_Q \vartheta u \, \mathrm{d}x\,\mathrm{d}t$$

by the properties of the maximal monotone operator α. Therefore, the quadruple $(\chi, \vartheta, u, \xi)$ fulfils (33)–(34), as well as (11) on behalf of (112) and (114).

The convergences (111)–(114) for the approximate solutions $\{\vartheta^\nu, \chi^\nu\}$ hold in fact along the whole sequence, since the limit pair (ϑ, x) is unique, as we will show below. In turn, the selections $\xi \in \beta(\chi)$ and $u \in \alpha(\vartheta)$ satisfying (33), (34) are not uniquely determined, and we cannot therefore reinforce the convergences (115)–(116). □

3.4. Uniqueness for Problem 1

Remark 3.4

We claim that there exists a positive constant C^*, depending only on T, $|\Omega|$ and L, such that for *any* solution pair (ϑ, χ) of Problem 1 corresponding to data ϑ_0, χ_0 and f which satisfy

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(30)–(32), there holds

$$\|\hat{\alpha}(\vartheta)\|_{L^\infty(0,T;L^1(\Omega))} + \|\partial_t\chi\|^2_{L^2(0,T;H)} + \|\chi\|^2_{L^\infty(0,T;V)} + \|\hat{\beta}(\chi)\|_{L^\infty(0,T;L^1(\Omega))}$$
$$\leqslant C^* \left(1 + \|\hat{\alpha}(\vartheta_0)\|_{L^1(\Omega)} + \|\hat{\beta}(\chi_0)\|_{L^1(\Omega)} + \|\chi_0\|^2_V + \|f\|^2_{L^2(0,T;V')}\right) \quad (127)$$

To check this, we test (33) by u, (34) by $\partial_t\chi$, integrate both on $(0,t)$ and add the resulting equations. Formal computations (which can be made rigorous by performing the regularization on $\partial_t\chi$ we have mentioned in the proof of Proposition 3.3) allow us to obtain, upon cancellation of two terms,

$$\int_\Omega \hat{\alpha}(\vartheta(x,t))\,\mathrm{d}x + \int_0^t |\nabla u(s)|^2_H\,\mathrm{d}s + \int_0^t |\partial_t\chi(s)|^2_H\,\mathrm{d}s + \frac{1}{2}|\nabla\chi(t)|^2_H$$
$$+ \int_\Omega \hat{\beta}(\chi(x,t))\,\mathrm{d}x \leqslant \int_0^t \langle f(s), u(s)\rangle\,\mathrm{d}s + \int_\Omega \hat{\alpha}(\vartheta_0(x))\,\mathrm{d}x + \frac{1}{2}\,|\nabla\chi^0\|^2_H$$
$$+ \int_\Omega \hat{\beta}(\chi^0(x))\,\mathrm{d}x \int_0^t \left(\int_\Omega |\sigma'(\chi(x,s))\partial_t\chi(x,s)|\,\mathrm{d}x\right)\mathrm{d}s$$

The first summand on the right-hand side of the above inequality can be dealt with by developing exactly the same estimates as in (117)–(123), while the last term can be treated arguing as in (102)–(103). Recalling the computations developed throughout the proof of Proposition 3.3, it is therefore easy to deduce that the estimate (127) holds.

Proof of Proposition 2.1
Let $(\chi^i, \vartheta^i, u^i, \xi^i)$, $i = 1, 2$ be two quadruples fulfilling the variational formulation of Problem 1 corresponding to the data ϑ_0^i, χ_0^i, and f^i and let $e^i := \vartheta^i + \chi^i$ be the corresponding *internal energies*. Let us put

$$\bar{\chi} := \chi^1 - \chi^2, \quad \bar{\vartheta} := \vartheta^1 - \vartheta^2, \quad \bar{e} := e^1 - e^2, \quad \bar{u} := u^1 - u^2, \quad \bar{\xi} := \xi^1 - \xi^2, \quad \bar{f} := f^1 - f^2$$

Clearly, the quadruple $(\bar{e}, \bar{\chi}, \bar{u}, \bar{\xi})$ satisfies

$$\partial_t\bar{e} + A\bar{u} = \bar{f} \quad \text{in } V', \quad \text{a.e. in } (0,T) \quad (128)$$

$$\partial_t\bar{\chi} + A\bar{\chi} + \bar{\xi} + \sigma'(\chi_1) - \sigma'(\chi_2) = \bar{u} \quad \text{in } H, \quad \text{a.e. in } (0,T) \quad (129)$$

Now (128) yields that

$$\frac{\mathrm{d}}{\mathrm{d}t}(m(\bar{e}(t))) = m(\bar{f}(t)) \quad \text{for a.e. } t \in (0,T) \quad (130)$$

whence, integrating in time, we find that there holds

$$|m(\bar{e}(t))| \leqslant C(\|\vartheta_0^1 - \vartheta_0^2\|_{V'} + \|\chi_0^1 - \chi_0^2\|_{V'} + \|f^1 - f^2\|_{L^2(0,T;V')}) \quad (131)$$

for a.e. $t \in (0,T)$ and for a positive constant C independent of t.

Let us test (128) by $\mathcal{N}(\bar{e}(t)-m(\bar{e}(t)))$, (129) by $\bar{\chi}(t)-m(\bar{e}(t))$, add the resulting equations and integrate in time: we have to take into account the cancellation of the terms $\int_0^t(\bar{u}(s),\bar{\chi}(s)-m(\bar{e}(s)))_H\,\mathrm{d}s$, and note that

$$\langle\partial_t\bar{e}(t),\mathcal{N}(\bar{e}(t)-m(\bar{e}(t)))\rangle=\langle\partial_t(\bar{e}(t)-m(\bar{e}(t))),\mathcal{N}(\bar{e}(t)-m(\bar{e}(t)))\rangle$$
$$=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{e}(t)-m(\bar{e}(t))\|_{V'}^2\quad\text{for a.e. } t\in(0,T) \tag{132}$$

since $\mathcal{N}(\bar{e}(t)-m(\bar{e}(t)))\in\mathcal{V}$ (the subspace of the zero mean value functions of V), taking into account (25) as well. Therefore, we get

$$\begin{aligned}&\frac{1}{2}\|\bar{e}(t)-m(\bar{e}(t))\|_{V'}^2+\int_0^t(\bar{u}(s),\bar{\vartheta}(s))_H\,\mathrm{d}s+\frac{1}{2}|\bar{\chi}(t)|_H^2+\int_0^t|\nabla(\bar{\chi}(s))|_H^2\,\mathrm{d}s\\&\quad-\int_0^t(\partial_t\bar{\chi}(s),m(\bar{e}(s)))_H\,\mathrm{d}s+\int_0^t(\bar{\xi}(s),\bar{\chi}(s))_H\,\mathrm{d}s\leqslant\frac{1}{2}\|\bar{e}(0)-m(\bar{e}(0))\|_{V'}^2\\&\quad+\frac{1}{2}|\chi_0^1-\chi_0^2|_H^2+\frac{1}{2}\int_0^t\|\bar{e}(s)-m(\bar{e}(s))\|_{V'}^2\,\mathrm{d}s+\left|\int_0^t(\sigma'(\chi^1(s))-\sigma'(\chi^2(s)),\bar{\chi}(s))_H\,\mathrm{d}s\right|\\&\quad+\frac{1}{2}\|\bar{f}\|_{L^2(0,T;V')}^2+\int_0^t\int_\Omega m(\bar{e}(s))\bar{\xi}(x,s)\,\mathrm{d}x\,\mathrm{d}s\\&\quad+\int_0^t(m(\bar{e}(s)),\sigma'(\chi^1(s))-\sigma'(\chi^2(s)))_H\,\mathrm{d}s\end{aligned} \tag{133}$$

Now, the second (sixth, resp.) term on the left-hand side of the above inequality is positive thanks to the monotonicity of the graph α (β, resp.). As for the right-hand side of (133), we first of all note that the fourth term is easily estimated by $C\|\bar{\chi}\|_{L^2(0,t;H)}^2$ in view of (29), and we can deal with the last term in the same way, also reminding the estimate (131) for $m(\bar{e})$. Moreover, integrating by parts and recalling (130), we note that

$$\begin{aligned}&\int_0^t(\partial_t\bar{\chi}(s),m(\bar{e}(s)))_H\,\mathrm{d}s\\&=-\int_0^t(m(\bar{f}(s)),\bar{\chi}(s))_H\,\mathrm{d}s+\int_\Omega m(\bar{e}(t))\bar{\chi}(x,t)\,\mathrm{d}x\\&\quad-\int_\Omega m(\bar{e}(0))\bar{\chi}(x,0)\,\mathrm{d}x\leqslant\frac{1}{2}\|\bar{\chi}\|_{L^2(0,t;H)}^2+\frac{1}{4}|\bar{\chi}(t)|_H^2\\&\quad+C(\|\vartheta_0^1-\vartheta_0^2\|_{V'}^2+|\chi_0^1-\chi_0^2|_H^2+\|\bar{f}\|_{L^2(0,T;V')}^2)\end{aligned} \tag{134}$$

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Finally, we have

$$\int_0^t |m(\bar{e}(s))| \|\bar{\xi}(s)\|_{L^1(\Omega)} \, ds$$

$$\leqslant \int_0^t |m(\bar{e}(s))| (\|\xi^1(s)\|_{L^1(\Omega)} + \|\xi^2(s)\|_{L^1(\Omega)}) \, ds$$

$$\leqslant 2M_\beta |\Omega| \int_0^t |m(\bar{e}(s))| \, ds + M_\beta \int_0^t |m(\bar{e}(s))| (\|\hat{\beta}(\chi^1(s))\|_{L^1(\Omega)}$$

$$+ \|\hat{\beta}(\chi^2(s))\|_{L^1(\Omega)}) \, ds \qquad (135)$$

Obviously, the first summand on the right-hand side can be treated by means of (131), while the second term is estimated by the data ϑ_0^i, χ_0^i and f^i of the problem as prescribed by the preliminary a priori estimate (127). Collecting (133)–(135) and applying Gronwall's Lemma to $\|\bar{e}(t) - m(\bar{e}(t))\|_{V'}^2 + |\bar{\chi}(t)|_H^2$, we readily deduce the continuous dependence estimate (35).

4. ASYMPTOTIC ANALYSIS OF PROBLEM $\mathbf{P}_\varepsilon$ AS $\varepsilon \downarrow 0$

First of all, we give a sketch of the proof of Proposition 2.2, which extends the techniques developed in Reference [13, Theorem 2.1] in demonstrating an analogous continuous dependence result for the viscous Cahn–Hilliard equation (in the particular case of (7)). On the other hand, the computations are similar to the ones developed for the proof Proposition 2.1. Therefore, we will just outline the main steps of the argument, referring the reader to Reference [13] and to Proposition 2.1 for the details.

Outline of the proof of Proposition 2.2

Preliminarly, we show that there exists a positive constant C_*, depending only on T, $|\Omega|$ and the Lipschitz constant L of σ', such that for *any* solution $\chi \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)$ to Problem 2 corresponding to data χ_0 and f which satisfy (36)–(37), there holds

$$\|\partial_t \chi\|_{L^2(0,T;H)}^2 + \|\chi\|_{L^\infty(0,T;V)}^2 + \|\hat{\beta}(\chi)\|_{L^\infty(0,T;L^1(\Omega))}$$

$$\leqslant C_*(1 + \|\hat{\beta}(\chi_0)\|_{L^1(\Omega)} + \|\chi_0\|_V^2 + \|f\|_{L^2(0,T;V')}^2) \qquad (136)$$

To see (136), we test (38) by $\mathcal{N}(\partial_t \chi)$ (noting that $m(\partial_t \chi) = 0$ by (37) and (38)), we multiply (39) by $\partial_t \chi$, we integrate the resulting equations on $(0,t)$ and we finally sum them up: the same computations of Remark 3.4 apply to this case, with obvious differences, and we get (136).

Now, to prove (40), we consider two solutions χ_1 and χ_2 to Problem 2 corresponding to the data χ_{0i}, f_i, $i=1,2$ and we let $\underline{\chi} := \chi_1 - \chi_2$. In view of the assumption (37) on the source terms f_i, we see that $\underline{\chi}$ has constant mean value $m(\underline{\chi}) = m(\chi_{01}) - m(\chi_{02})$. Then, we subtract equation (38) for χ_2 from (38) for χ_1, and we analogously proceed for (39); we test the equations thus obtained by $\mathcal{N}(\underline{\chi} - m(\underline{\chi}))$, and by $\underline{\chi} - m(\underline{\chi})$, respectively. Next, we integrate in time and add the ensuing relations. Therefore, we get

$$
\begin{aligned}
&\frac{1}{2}\,\|\underline{\chi}(t) - m(\underline{\chi})\|_{V'}^2 + \frac{1}{2}\,|\underline{\chi}(t) - m(\underline{\chi})|_H^2 + \int_0^t |\nabla\underline{\chi}(s)|_H^2\,\mathrm{d}s \\
&\quad + \int_0^t (\underline{\xi}(s), \underline{\chi}(s))_H\,\mathrm{d}s \leqslant \frac{1}{2}\,\|\chi_{01} - \chi_{02} - m(\underline{\chi})\|_{V'}^2 + \frac{1}{2}\,|\chi_{01} - \chi_{02} - m(\underline{\chi})|_H^2 \\
&\quad + \frac{1}{2}\|f_1 - f_2\|_{L^2(0,T;V')}^2 + \frac{1}{2}\,\|\underline{\chi} - m(\underline{\chi})\|_{L^2(0,T;V')}^2 \\
&\quad + \left|\int_0^t (\sigma'(\chi_1(s)) - \sigma'(\chi_2(s)), \underline{\chi}(s) - m(\underline{\chi}))_H\,\mathrm{d}s\right| + \int_0^t (m(\underline{\chi}), \xi_1(s) - \xi_2(s))_H\,\mathrm{d}s
\end{aligned}
$$

Comparing the above inequality with (133) in the proof of Proposition 2.1, we see that the last two summands can be treated by performing the same estimates developed therein. In particular, we have to make use of the a priori bound (136) for $\|\hat{\beta}(\chi_i)\|_{L^\infty(0,T;L^1(\Omega))}$ in terms of the data χ_{0i} and f_i, $i=1,2$. Thus, we succeed in obtaining (40). □

Proof of (i) in Theorem 2

In order to obtain *a priori* estimates on the sequences of approximate solutions $\{\chi_\varepsilon\}$ and $\{\vartheta_\varepsilon\}$, first of all we test (33)—in which the maximal monotone graph α is given by the function α_ε in (17)—by $u_\varepsilon = \alpha_\varepsilon(\vartheta_\varepsilon) = \varepsilon^\eta\vartheta_\varepsilon - 1/\vartheta_\varepsilon$ and we integrate in time. Next, we multiply (34) by $\partial_t\chi_\varepsilon$, and integrate on $(0,t)$. We add the resulting equations; two terms cancel out, and, upon performing the regularization on the graph α_ε described in the previous Section 3.2, as well as regularizing the test function $\partial_t\chi_\varepsilon$, we obtain

$$
\begin{aligned}
&\frac{\varepsilon^{1+\eta}}{2}\,|\vartheta_\varepsilon(t)|_H^2 - \varepsilon\int_\Omega \log(\vartheta_\varepsilon(x,t))\,\mathrm{d}x + \int_0^t |\partial_t\chi_\varepsilon(s)|_H^2\,\mathrm{d}s + \int_0^t |\nabla(u_\varepsilon(s))|_H^2\,\mathrm{d}s \\
&\quad + \frac{1}{2}\,|\nabla(\chi_\varepsilon(t))|_H^2 + \int_\Omega \hat{\beta}(\chi_\varepsilon(x,t))\,\mathrm{d}x + \int_0^t (\sigma'(\chi_\varepsilon(s)), \partial_t\chi_\varepsilon(s))_H\,\mathrm{d}s = \frac{1}{2}\,|\nabla(\chi_\varepsilon^0)|_H^2 \\
&\quad + \int_\Omega \hat{\beta}(\chi_\varepsilon^0(x))\,\mathrm{d}x + \frac{\varepsilon^{1+\eta}}{2}\,|\vartheta_\varepsilon^0|_H^2 - \varepsilon\int_\Omega \log(\vartheta_\varepsilon^0(x))\,\mathrm{d}x + \int_0^t \langle f_\varepsilon(s), u_\varepsilon(s)\rangle\,\mathrm{d}s
\end{aligned} \tag{137}
$$

Let us recall the elementary inequality $r - \log(r) \geqslant \frac{1}{3}(r + |\log(r)|)$ for every $r > 0$, entailing

$$\begin{aligned}\frac{\varepsilon^{1+\eta}}{2}|\vartheta_\varepsilon(t)|_H^2-\varepsilon\int_\Omega\log(\vartheta_\varepsilon(x,t))\,\mathrm{d}x &\geqslant \frac{\varepsilon^{1+\eta}}{2}|\vartheta_\varepsilon(t)|_H^2-\varepsilon\int_\Omega|\vartheta_\varepsilon(x,t)|\,\mathrm{d}x\\ &\geqslant \frac{\varepsilon^{1+\eta}}{2}|\vartheta_\varepsilon(t)|_H^2-\frac{\varepsilon^2}{4}|\vartheta_\varepsilon(t)|_H^2-C\\ &\geqslant \frac{\varepsilon^{1+\eta}}{4}|\vartheta_\varepsilon(t)|_H^2-C \end{aligned}\tag{138}$$

where we have used the elementary fact that $\varepsilon^2\leqslant\varepsilon^{1+\eta}$ for $\varepsilon\leqslant 1$ and for every $0<\eta<1$. The last integral on the left-hand side of (137) can be handled as in (102)–(103), while for the last summand on the right-hand side we argue exactly in the same way as in the proof of Proposition 3.3—namely exploiting (28) and applying Gronwall's Lemma, see (119)–(123). Collecting (137) and (138), in view of the assumptions (44)–(45) on the approximate data, we easily infer that there exists a positive constant C independent of ε such that

$$\begin{aligned}&\varepsilon^{(1+\eta)/2}\|\vartheta_\varepsilon\|_{L^\infty(0,T;H)}+\varepsilon\|\log(\vartheta_\varepsilon)\|_{L^\infty(0,T;L^1(\Omega))}+\|\chi_\varepsilon\|_{H^1(0,T;H)\cap L^\infty(0,T;V)}\\ &\quad+\|\hat\beta(\chi_\varepsilon)\|_{L^\infty(0,T;L^1(\Omega))}+\|u_\varepsilon\|_{L^2(0,T;V)}\leqslant C\end{aligned}\tag{139}$$

(notice that the estimate for $\|u_\varepsilon\|_{L^2(0,T;V)}$ follows from the bound for $\|\nabla u_\varepsilon\|_{L^2(0,T;H)}$, as well as from the estimate for $\|m(u_\varepsilon)\|_{L^2(0,T)}$, which can be inferred by comparison in (34)). Proceeding in the same way as in the proof of Propositions 3.2 and 3.3, we obtain the further estimates

$$\|\chi_\varepsilon\|_{L^2(0,T;W)}+\|\xi_\varepsilon\|_{L^2(0,T;H)}+\varepsilon\|\partial_t\vartheta_\varepsilon\|_{L^2(0,T;V')}\leqslant C\quad\forall\varepsilon>0\tag{140}$$

Therefore, (48) immediately follows, while invoking the (weak) compactness results of Reference [31, Theorem 4, Cor. 5], we find that there exists a triplet (χ,u,ξ) such that the convergences (46)–(47) and (49)–(50) hold along some subsequence $\{\varepsilon_k\}$. Thanks to (42) as well, we can pass to the limit in the (variational formulation for the) approximate equations (15) and (16) and we find that χ, u, and ξ fulfil (38) and (39)—indeed to see that $\xi\in\beta(\chi)$ a.e. in Q it suffices to argue by monotonicity as for (126). Finally, (47) and (41) ensure that χ satisfies the initial condition (8), and it is thus the unique solution of Problem 2; by uniqueness, we infer that the convergences (46)–(47) hold in fact along the whole sequences $\{\chi_\varepsilon\}$ and $\{\vartheta_\varepsilon\}$.

Proof of (ii) in Theorem 2
Let us set

$$\tilde\chi_\varepsilon:=\chi_\varepsilon-\chi,\quad \tilde e_\varepsilon:=\varepsilon\vartheta_\varepsilon+\chi_\varepsilon-\chi,\quad \tilde u_\varepsilon:=u_\varepsilon-u\quad \tilde\xi_\varepsilon:=\xi_\varepsilon-\xi,\quad \tilde f_\varepsilon:=f_\varepsilon-f$$

Subtracting (38) and (39) for (χ, u, ξ) from (15) and (16), respectively, for $(\chi_\varepsilon, \vartheta_\varepsilon, u_\varepsilon, \xi_\varepsilon)$, we find

$$\partial_t \tilde{e}_\varepsilon + A\tilde{u}_\varepsilon = \tilde{f}_\varepsilon, \quad \text{in } V', \quad \text{a.e. in } (0,T) \tag{141}$$

$$\partial_t \tilde{\chi}_\varepsilon + A\tilde{\chi}_\varepsilon + \tilde{\xi}_\varepsilon + \sigma'(\chi_\varepsilon) - \sigma'(\chi) = \tilde{u}_\varepsilon, \quad \text{in } H, \quad \text{a.e. in } (0,T) \tag{142}$$

Arguing as in the proof of Proposition 2.1, we see that

$$|m(\tilde{e}_\varepsilon(t))| \leqslant C(\varepsilon\|\vartheta_\varepsilon^0\|_{V'} + \|\chi_\varepsilon^0 - \chi_0\|_{V'} + \|\tilde{f}_\varepsilon\|_{L^2(0,T;V')}) \tag{143}$$

Let us test (141) by $\mathcal{N}(\tilde{e}_\varepsilon(t) - m(\tilde{e}_\varepsilon(t)))$, (142) by $\tilde{\chi}_\varepsilon(t) - m(\tilde{e}_\varepsilon(t))$, add the resulting equations and integrate in time. In view of (132) and cancelling two terms like for (133) in the proof of Proposition 2.1, we conclude

$$\begin{aligned}
&\frac{1}{2}\|\tilde{e}_\varepsilon(t) - m(\tilde{e}_\varepsilon(t))\|_{V'}^2 + \int_0^t (\varepsilon\vartheta_\varepsilon(s), \tilde{u}_\varepsilon(s))_H \,\mathrm{d}s + \int_0^t (\partial_t \tilde{\chi}_\varepsilon(s), m(\tilde{e}_\varepsilon(s)))_H \,\mathrm{d}s \\
&\quad + \frac{1}{2}|\tilde{\chi}_\varepsilon(t)|_H^2 + \int_0^t |\nabla(\tilde{\chi}_\varepsilon(s))|_H^2 \,\mathrm{d}s + \int_0^t (\sigma'(\chi_\varepsilon(s)) - \sigma'(\chi(s)), \tilde{\chi}_\varepsilon(s))_H \,\mathrm{d}s \\
&\quad + \int_0^t (\tilde{\xi}_\varepsilon(s), \tilde{\chi}_\varepsilon(s))_H \,\mathrm{d}s \leqslant \frac{1}{2}\|\tilde{e}_\varepsilon(0) - m(\tilde{e}_\varepsilon(0))\|_{V'}^2 + \frac{1}{2}|\chi_\varepsilon^0 - \chi_0|_H^2 + \frac{1}{2}\|\tilde{f}_\varepsilon\|_{L^2(0,T;V')}^2 \\
&\quad + \frac{1}{2}\int_0^t \|\tilde{e}_\varepsilon(s) - m(\tilde{e}_\varepsilon(s))\|_{V'}^2 \,\mathrm{d}s + \int_0^t (m(\tilde{e}_\varepsilon(s)), \tilde{\xi}_\varepsilon(s) + \sigma'(\chi_\varepsilon(s)) - \sigma'(\chi(s)))_H \,\mathrm{d}s
\end{aligned} \tag{144}$$

The only substantial difference between the inequality above and (133) in the proof of Proposition 2.1 consists in the second term on the left-hand side, which is bounded by

$$\|\varepsilon\vartheta_\varepsilon\|_{L^2(0,T;H)}\|\tilde{u}_\varepsilon\|_{L^2(0,T;H)} \leqslant C\varepsilon^{(1-\eta)/2} \tag{145}$$

on behalf of (139). As for the remaining summands, we can work out the analogues of the estimates (134) and (135); in particular, once again we will have to exhibit an a priori estimate for $\|\hat{\beta}(\chi_\varepsilon)\|_{L^1(\Omega)}$, which is readily inferred from (139). Therefore, upon recalling (143) for $m(\tilde{e}_\varepsilon(t))$ and applying Gronwall's Lemma to $\|\tilde{e}_\varepsilon(t) - m(\tilde{e}_\varepsilon(t))\|_{V'}^2 + |\tilde{\chi}_\varepsilon(t)|_H^2$, we easily derive from (144)

$$\begin{aligned}
&\|\tilde{e}_\varepsilon\|_{L^\infty(0,T;V')}^2 + \|\tilde{\chi}_\varepsilon\|_{C^0([0,T];H)\cap L^2(0,T;V)}^2 \\
&\quad \leqslant C(\varepsilon^2\|\vartheta_\varepsilon^0\|_{V'}^2 + \|\chi_\varepsilon^0 - \chi^0\|_{V'}^2 + \|\tilde{f}_\varepsilon\|_{L^2(0,T;V')}^2 + |\chi_\varepsilon^0 - \chi^0|_H^2 + \varepsilon^{(1-\eta)/2})
\end{aligned}$$

yielding (51) on account of (44) as well, whereas (52) obviously follows from (139). □

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5. ASYMPTOTIC ANALYSIS FOR $\mathbf{P}^*_{\varepsilon\delta}$ AS $\varepsilon, \delta \downarrow 0$

Proof of (i) in Theorem 4
Proceeding as in the proof of Theorem 2, we test (33) (with $\alpha = \alpha_\varepsilon$) by $u_{\varepsilon\delta} = \varepsilon^\eta \vartheta_{\varepsilon\delta} - 1/\vartheta_{\varepsilon\delta}$, the variational formulation (34) of (21) by $\partial_t \chi_{\varepsilon\delta}$, we sum up and integrate on $(0, t)$. Therefore we get the analogue of (137)

$$\frac{\varepsilon^{1+\eta}}{4} |\vartheta_{\varepsilon\delta}(t)|_H^2 + \delta \int_0^t |\partial_t \chi_{\varepsilon\delta}(s)|_H^2 \, \mathrm{d}s + \int_0^t |\nabla(u_{\varepsilon\delta}(s))|_H^2 \, \mathrm{d}s + \frac{1}{2} |\nabla(\chi_{\varepsilon\delta}(t))|_H^2$$
$$+ \int_\Omega \left(\frac{\chi_{\varepsilon\delta}^4(x,t)}{4} - \frac{\chi_{\varepsilon\delta}^2(x,t)}{2} \right) \mathrm{d}x \leqslant C + \frac{\varepsilon^{1+\eta}}{2} |\vartheta_{\varepsilon\delta}^0|_H^2 - \varepsilon \int_\Omega \log(\vartheta_{\varepsilon\delta}^0(x)) \, \mathrm{d}x$$
$$+ \frac{1}{2} |\nabla(\chi_{\varepsilon\delta}^0)|_H^2 + \int_\Omega (\chi_{\varepsilon\delta}^0(x))^4 \, \mathrm{d}x + \int_0^t \langle f_{\varepsilon\delta}(s), u_{\varepsilon\delta}(s) \rangle \, \mathrm{d}s \tag{146}$$

The last term on the left-hand side can be handled by means of (73), choosing e.g. $\rho = 1/2$. In turn, arguing as in (117)–(121), we see that for the last summand on the left-hand side it suffices to observe that

$$\int_0^t |\langle f_{\varepsilon\delta}(s), m(\delta \partial_t \chi_{\varepsilon\delta}(s) + \chi_{\varepsilon\delta}^3(s) - \chi_{\varepsilon\delta}(s)) \rangle \, \mathrm{d}s$$
$$\leqslant |\Omega|^{1/2} \int_0^t \| f_{\varepsilon\delta}(s) \|_{V'} |m(\delta \partial_t \chi_{\varepsilon\delta}(s) + \chi_{\varepsilon\delta}^3(s) - \chi_{\varepsilon\delta}(s))| \mathrm{d}s$$
$$\leqslant C(1 + \| f_{\varepsilon\delta} \|_{L^2(0,T;V')}^2) + \frac{\delta^2}{2} \int_0^t |\partial_t \chi_{\varepsilon\delta}(s)|_H^2 \, \mathrm{d}s$$
$$+ \int_0^t \| f_{\varepsilon\delta}(s) \|_{V'} \| \chi_{\varepsilon\delta}(s) \|_{L^4(\Omega)}^4 \, \mathrm{d}s \tag{147}$$

where we have applied (74) with $\mu = |\Omega|^{1/2}$. Combining (146) and (147), we can estimate the second summand on the right-hand side of the above inequality with the first term on the left-hand side of (146), since $\delta^2 \leqslant \delta$ for $\delta \leqslant 1$. Finally, we apply Gronwall's Lemma to $\| \chi_{\varepsilon\delta}(t) \|_{L^4(\Omega)}^4$ and we argue as in the proof of Theorem 2 (see (137)–(138)), so that we easily deduce

$$\varepsilon^{(1+\eta)/2} \| \vartheta_{\varepsilon\delta} \|_{L^2(0,T;H)} + \| \chi_{\varepsilon\delta} \|_{L^2(0,T;W) \cap L^\infty(0,T;V)} + \delta^{1/2} \| \partial_t \chi_{\varepsilon\delta} \|_{L^2(0,T;H)}$$
$$+ \| \varepsilon \partial_t \vartheta_{\varepsilon\delta} + \partial_t \chi_{\varepsilon\delta} \|_{L^2(0,T;V')} + \| u_{\varepsilon\delta} \|_{L^2(0,T;V)} \leqslant C \tag{148}$$

for a constant $C \geqslant 0$ independent of ε and δ. Hence, we readily find that (67) and (68) hold; further, by the compactness results in Reference [31], there exist χ and u such that (64) and

(66) hold along some subsequence as ε and $\delta \downarrow 0$. Now, taking into account (67) as well, we deduce (69) for the same subsequence. The estimate (148) and the results in Reference [31] ensure that $\{\varepsilon\vartheta_{\varepsilon\delta} + \chi_{\varepsilon\delta}\}$ strongly converges in $C^0([0,T];V')$, so that we infer the first part of (65).

The second one is instead deduced from

$$\|\chi_{\varepsilon\delta} - \chi\|^2_{L^2(0,T;V)} \leqslant \|\chi_{\varepsilon\delta} - \chi\|^2_{L^2(0,T;V')} \|\chi_{\varepsilon\delta} - \chi\|^{1/2}_{L^2(0,T;V)} \|\chi_{\varepsilon\delta} - \chi\|_{L^2(0,T;W)}$$

which follows from well-known interpolation results: note that the first factor on the right-hand side vanishes as ε and $\delta \downarrow 0$, whereas the last one is bounded in view of (148). Since $\chi_{\varepsilon\delta} \to \chi$ a.e. in Q along a subsequence by the Egorov theorem, by a simple application of the dominated convergence theorem we conclude that

$$\chi^3_{\varepsilon\delta} \to \chi^3 \quad \text{in } L^2(0,T;H) \quad \text{as } \varepsilon \text{ and } \delta \downarrow 0 \tag{149}$$

Finally, the further regularity $\chi \in C^0([0,T];H)$ ensues from $\chi \in L^\infty(0,T;V) \cap C^0([0,T];V')$, which gives $\chi \in C^0_w([0,T];V) \subset C^0([0,T];H)$ on behalf of [34, Lemma III.1.4].

By the convergences proved so far, we manage to pass to the limit in (15) and (21) and we achieve that the limit pair (χ,u) fulfils (59)–(60) with the initial condition (8). This done, we deduce that χ is the unique solution to Problem 3; since the limit u is uniquely determined as well by (60), we recover that (64)–(66) and (69) hold for the whole sequence. □

Proof of (ii) in Theorem 4

We introduce $\tilde{\chi}_{\varepsilon\delta} := \chi_{\varepsilon\delta} - \chi$, $\tilde{e}_{\varepsilon\delta} := \varepsilon\vartheta_{\varepsilon\delta} + \tilde{\chi}_{\varepsilon\delta}$, $\tilde{u}_{\varepsilon\delta} := u_{\varepsilon\delta} - u$, and $\widetilde{f_{\varepsilon\delta}} := f_{\varepsilon\delta} - f$; in this framework, an estimate analogous to (143) holds for $m(\tilde{e}_{\varepsilon\delta})$, and, on account of (148), we further have

$$\begin{aligned}|m(\tilde{\chi}_{\varepsilon\delta}(t))| &\leqslant \varepsilon\,|m(\vartheta_{\varepsilon\delta}(t))| + |m(\tilde{e}_{\varepsilon\delta}(t))| \\ &\leqslant \varepsilon^{(1-\eta)/2}\|\vartheta_{\varepsilon\delta}\|_{L^2(0,T;H)} + C(\varepsilon\|\vartheta^0_{\varepsilon\delta}\|_{V'} + \|\chi^0_{\varepsilon\delta} - \chi^0\|_{V'} + \|\widetilde{f_{\varepsilon\delta}}\|_{L^2(0,T;V')})\end{aligned} \tag{150}$$

Let us take the difference between (15) and (21) for the pair $(\chi_{\varepsilon\delta}, \vartheta_{\varepsilon\delta})$ and (59) and (60) for (χ,u). We obtain that $(\tilde{\chi}_{\varepsilon\delta}, \tilde{e}_{\varepsilon\delta}, \tilde{u}_{\varepsilon\delta})$ satisfy the analogue of (141), as well as

$$\delta\partial_t\chi_{\varepsilon\delta} + A\tilde{\chi}_{\varepsilon\delta} + \chi^3_{\varepsilon\delta} - \chi^3 - (\chi_{\varepsilon\delta} - \chi) = \tilde{u}_{\varepsilon\delta} \quad \text{in } V', \quad \text{a.e. in } (0,T) \tag{151}$$

We test the former equation by $\tilde{e}_{\varepsilon\delta} - m(\tilde{e}_{\varepsilon\delta})$, the latter one by $\tilde{\chi}_{\varepsilon\delta} - m(\tilde{e}_{\varepsilon\delta})$, we integrate in time and sum up. We thus get an inequality perfectly analogous to (144)

$$\begin{aligned}&\frac{1}{2}\|\tilde{e}_{\varepsilon\delta}(t) - m(\tilde{e}_{\varepsilon\delta}(t))\|^2_{V'} + \int_0^t (\varepsilon\vartheta_{\varepsilon\delta}(s), \tilde{u}_{\varepsilon\delta}(s))_H\,\mathrm{d}s + \int_0^t |\nabla(\tilde{\chi}_{\varepsilon\delta}(s))|^2_H\,\mathrm{d}s \\ &\quad + \delta\int_0^1 (\partial_t\chi_{\varepsilon\delta}(s), \tilde{\chi}_{\varepsilon\delta}(s) - m(\tilde{e}_{\varepsilon\delta}(s)))_H\,\mathrm{d}s + \int_0^t (\chi^3_{\varepsilon\delta}(s) - \chi^3(s), \tilde{\chi}_{\varepsilon\delta}(s))_H\,\mathrm{d}s \\ &\leqslant \frac{1}{2}\|\tilde{e}_{\varepsilon\delta}(0) - m(\tilde{e}_{\varepsilon\delta}(0))\|^2_{V'} + \frac{1}{2}\|\widetilde{f_{\varepsilon\delta}}\|^2_{L^2(0,T;V')} + \frac{1}{2}\int_0^t \|\tilde{e}_{\varepsilon\delta}(s) - m(\tilde{e}_{\varepsilon\delta}(s))\|^2_{V'}\,\mathrm{d}s \\ &\quad + \int_0^t (\tilde{\chi}_{\varepsilon\delta}(s), \tilde{\chi}_{\varepsilon\delta}(s) - m(\tilde{e}_{\varepsilon\delta}(s)))_H\,\mathrm{d}s + \int_0^1 (m(\tilde{e}_{\varepsilon\delta}(s)), \chi^3_{\varepsilon\delta}(s) - \chi^3(s))_H\,\mathrm{d}s\end{aligned} \tag{152}$$

Then, we recall the estimate (145) for the second term on the left-hand side, while the fifth integral is positive by monotonicity; as for the fourth summand, we can rewrite it as

$$\frac{\delta}{2}\int_0^t \frac{\mathrm{d}}{\mathrm{d}t}|\tilde{\chi}_{\varepsilon\delta}(s)-m(\tilde{e}_{\varepsilon\delta}(s))|_H^2\,\mathrm{d}s+\delta\int_0^t(\partial_t\chi(s),\tilde{\chi}_{\varepsilon\delta}(s)-m(\tilde{e}_{\varepsilon\delta}(s)))_H\,\mathrm{d}s$$
$$+\delta\int_0^t(m(\widetilde{f_{\varepsilon\delta}}(s)),\tilde{\chi}_{\varepsilon\delta}(s)-m(\tilde{e}_{\varepsilon\delta}(s)))_H\,\mathrm{d}s \tag{153}$$

also taking into account that $(\mathrm{d}/\mathrm{d}t)m(\tilde{e}_{\varepsilon\delta}(t))=m(\widetilde{f_{\varepsilon\delta}}(t))$ for a.e. $t\in(0,T)$. Clearly, the first term in (153) brings about a bound for $\delta^{1/2}\|\tilde{\chi}_{\varepsilon\delta}-m(\tilde{e}_{\varepsilon\delta})\|_{L^\infty(0,T;H)}$, while we can estimate the two remaining summands by

$$\frac{\delta^2}{2}\|\partial_t\chi\|^2_{L^2(0,T;H)}+\frac{\delta^2}{2}\|\widetilde{f_{\varepsilon\delta}}\|^2_{L^2(0,T;V')}+\|\tilde{\chi}_{\varepsilon\delta}-m(\tilde{e}_{\varepsilon\delta})\|^2_{L^2(0,t;H)} \tag{154}$$

By suitably rearranging the terms, we see that the fourth integral on the right-hand side of (152) equals

$$\int_0^t\langle\tilde{e}_{\varepsilon\delta}(s),\tilde{\chi}_{\varepsilon\delta}(s)-m(\tilde{\chi}_{\varepsilon\delta}(s))\rangle\,\mathrm{d}s-\int_0^t\langle\tilde{e}_{\varepsilon\delta}(s),m(\varepsilon\vartheta_{\varepsilon\delta}(s))\rangle\,\mathrm{d}s$$
$$-\int_0^t\langle\varepsilon\vartheta_{\varepsilon\delta}(s),\tilde{\chi}_{\varepsilon\delta}(s)-m(\tilde{\chi}_{\varepsilon\delta}(s))\rangle\,\mathrm{d}s+\int_0^t\langle\varepsilon\vartheta_{\varepsilon\delta}(s),m(\varepsilon\vartheta_{\varepsilon\delta}(s))\rangle\,\mathrm{d}s$$
$$\leqslant\frac{1}{2}\|\tilde{\chi}_{\varepsilon\delta}-m(\tilde{\chi}_{\varepsilon\delta})\|^2_{L^2(0,t;V)}+C(\|\tilde{e}_{\varepsilon\delta}\|^2_{L^2(0,t;V')}+\varepsilon^2\|\vartheta_{\varepsilon\delta}\|^2_{L^2(0,T;V')})$$

We can estimate the second summand on the right-hand side of the above inequality by $\|\tilde{e}_{\varepsilon\delta}(t)-m(\tilde{e}_{\varepsilon\delta}(t))\|^2_{V'}+\|m(\tilde{e}_{\varepsilon\delta}(t))\|^2_{V'}$, also reminding the bound (150) for $\|m(\tilde{e}_{\varepsilon\delta})\|_{L^\infty(0,T)}$, $\|m(\tilde{e}_{\varepsilon\delta})\|$, while by Poincaré's inequality the $L^\infty(0,T)$ first integral term is estimated. Note also that by Poincaré's inequality we can estimate the first integral term on the right-hand side of the above inequality by means of the term $\int_0^t|\nabla(\tilde{\chi}_{\varepsilon\delta}(s))|_H^2\,\mathrm{d}s$ on the left-hand side of (152). Finally, referring to Reference [13, Proof of Theorems 2.1, 2.6] for the details, we observe that for the last term of (152) there holds

$$\int_0^t|m(\tilde{e}_{\varepsilon\delta}(s))||\chi^3_{\varepsilon\delta}(s)-\chi^3(s)|_H\,\mathrm{d}s$$
$$\leqslant\int_0^t|m(\tilde{e}_{\varepsilon\delta}(s))||\langle\chi^2_{\varepsilon\delta}(s)+\chi^2(s)+\chi_{\varepsilon\delta}(s)\chi(s),\tilde{\chi}_{\varepsilon\delta}(s)-m(\tilde{\chi}_{\varepsilon\delta}(s))\rangle\,\mathrm{d}s$$
$$+\int_0^t|m(\tilde{e}_{\varepsilon\delta}(s))||m(\tilde{\chi}_{\varepsilon\delta}(s))||\langle\chi^2_{\varepsilon\delta}(s)+\chi^2(s)+\chi_{\varepsilon\delta}(s)\chi(s),1\rangle|\,\mathrm{d}s\leqslant\frac{1}{4}\|\tilde{\chi}_{\varepsilon\delta}-m(\tilde{\chi}_{\varepsilon\delta})\|^2_{L^2(0,t;V)}$$
$$+C(\|m(\tilde{e}_{\varepsilon\delta})\|^2_{L^\infty(0,T)}+\|m(\tilde{e}_{\varepsilon\delta})\|_{L^\infty(0,T)}\|m(\tilde{\chi}_{\varepsilon\delta})\|_{L^\infty(0,T)})(1+\|\chi_{\varepsilon\delta}\|^4_{L^4(Q)}+\|\chi\|^4_{L^4(Q)}) \tag{155}$$

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Collecting (152)–(157), taking into account (150), the *a priori* estimates (148), the assumption (44) on the initial data $\vartheta^0_{\varepsilon\delta}$, and finally invoking Gronwall's Lemma for the term $\|\tilde{e}_{\varepsilon\delta}(t) - m(\tilde{e}_{\varepsilon\delta}(t))\|^2_{V'}$ in (152), we finally obtain

$$\|\tilde{e}_{\varepsilon\delta}\|^2_{C^0([0,T];V')} + \delta\|\tilde{\chi}_{\varepsilon\delta}\|^2_{L^\infty(0,T;H)} + \|\tilde{\chi}_{\varepsilon\delta} - m(\tilde{\chi}_{\varepsilon\delta})\|^2_{L^2(0,T;V)}$$
$$\leqslant C(\varepsilon^{(1-\eta)/2} + \delta^2 + \|\widetilde{f_{\varepsilon\delta}}\|^2 + \|\chi^0_{\varepsilon\delta} - \chi^0\|^2_{V'} + \delta|\chi^0_{\varepsilon\delta} - \chi^0|^2_H)$$

which yields (70). □

Proof of Theorem 3
Preliminarily, we test (54) by $1-1/\vartheta^\varepsilon_*$, (55) by $\partial_t\chi^\varepsilon_*$, integrate in time both equations and sum them up. By performing the same computations as in the proof of Theorem 2, we recover the *a priori* bound

$$\|\chi^\varepsilon_*\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} + \|u^\varepsilon_*\|_{L^2(0,T;V)} + \|\xi^\varepsilon_*\|_{L^2(0,T;H)} \leqslant C \tag{156}$$

for the approximate solutions χ^ε_*, u^ε_*; note that we do not however dispose of estimates on ϑ^ε_*, which prevents us from directly passing to the limit in the approximate system (54)–(55).

Next, we integrate both (54) for $(\vartheta^\varepsilon_*,\ \chi^\varepsilon_*)$ and (38) for (χ, u) on $(0,t)$, $t \in (0,T)$, we subtract one from another, and we eventually get

$$\varepsilon\vartheta^\varepsilon_*(t) + \chi^\varepsilon_*(t) - \chi(t) + A\left(\int_0^t (u^\varepsilon_*(s) - u(s))\,\mathrm{d}s\right)$$
$$= \int_0^t (f^\varepsilon_*(s) - f(s))\,\mathrm{d}s + \varepsilon\vartheta^0_{*\varepsilon} + \chi^0_{*\varepsilon} - \chi_0 \quad \text{in } V'$$

which we test by $u^\varepsilon_*(t) - u(t)$ for a.e. $t \in (0,t)$. Then we take the difference between (55) and (39) and we test it by $\chi^\varepsilon_* - \chi$. We add the two resulting equations and integrate in time, which yields

$$\varepsilon\int_0^t \langle\vartheta^\varepsilon_*(s), u^\varepsilon_*(s) - u(s)\rangle\,\mathrm{d}s + \frac{1}{2}|\chi^\varepsilon_*(t) - \chi(t)|^2_H + \int_0^t |\nabla(\chi^\varepsilon_*(s) - \chi(s))|^2_H\,\mathrm{d}s$$
$$+ \int_0^t \left(\nabla\left(\int_0^s (u^\varepsilon_*(r) - u(r))\,\mathrm{d}r\right),\ \nabla(u^\varepsilon_*(s) - u(s))\right)_H \mathrm{d}s$$
$$+ \int_0^t (\xi^\varepsilon_*(s) - \xi(s), \chi^\varepsilon_*(s) - \chi(s))_H\,\mathrm{d}s = \int_0^t \left\langle \int_0^s (f^\varepsilon_*(r) - f(r))\,\mathrm{d}r,\ u^\varepsilon_*(s) - u(s)\right\rangle \mathrm{d}s$$
$$+ \varepsilon\int_0^t \langle\vartheta^0_{*\varepsilon}, u^\varepsilon_*(s) - u(s)\rangle\,\mathrm{d}s + \int_0^t (\chi^0_{*\varepsilon}(s) - \chi_0(s), u^\varepsilon_*(s) - u(s))_H\,\mathrm{d}s$$
$$- \int_0^t (\sigma'(\chi^\varepsilon_*(s)) - \sigma'(\chi(s)), \chi^\varepsilon_*(s) - \chi(s))_H \tag{157}$$

The first term on the left-hand side can be split into

$$\varepsilon \int_0^t \langle \vartheta_*^\varepsilon(s) - \vartheta(s), u_*^\varepsilon(s) - u(s) \rangle \, \mathrm{d}s + \varepsilon \int_0^t \langle \vartheta(s), u_*^\varepsilon(s) - u(s) \rangle \, \mathrm{d}s$$

ϑ fulfilling (56), so that the first summand is positive by monotonicity, whereas the second one is estimated by $\varepsilon \|\vartheta\|_{L^2(0,T;V')} \|u_*^\varepsilon - u\|_{L^2(0,T;V)} \to 0$ as $\varepsilon \downarrow 0$, thanks to (156). Similarly, we estimate the first summand on the right-hand side of (157) by

$$\int_0^t \left\| \int_0^s (f_*^\varepsilon(r) - f(r)) \, \mathrm{d}r \right\|_{V'} \|u_*^\varepsilon(s) - u(s)\|_V \, \mathrm{d}s \leqslant C \|f_*^\varepsilon - f\|_{L^2(0,T;V')} \|u_*^\varepsilon - u\|_{L^2(0,T;V)}$$

which vanishes for $\varepsilon \downarrow 0$ by (42). The next two terms on the right-hand side of (157) can be estimated likewise, on behalf of the assumptions (41) and (53) on the approximate data; finally, the last integral is treated once again by exploiting the Lipschitz continuity of σ'. Applying Gronwall's Lemma to $|\chi_*^\varepsilon(t) - \chi(t)|_H^2$, we infer from (157) and (156) that (57) holds, as well as

$$\nabla \left(\int_0^t (u_*^\varepsilon(s) - u(s)) \, \mathrm{d}s \right) \to 0 \quad \text{in } L^\infty(0,T;H^N) \text{ as } \varepsilon \downarrow 0 \tag{158}$$

On the other hand, (156) ensures that u_*^ε is weakly converging in $L^2(0,T;V)$, at least along a subsequence; clearly, for any subsequence $\{u^{\varepsilon_k}\}$ weakly converging in $L^2(0,T;V)$ to some limit u^*, there holds $\nabla(u^{\varepsilon_k}) \rightharpoonup \nabla(u^*)$ in $L^2(0,T;H^N)$. By comparison with (158), we conclude that

$$\int_0^t \nabla u(s) \, \mathrm{d}s = \int_0^t \nabla u^*(s) \, \mathrm{d}s \quad \text{for a.e. } t \in (0,T)$$

being t arbitrary, (58) ensues. □

Remark 5.1

It is clear from the argument developed above that (58) can be reinforced to

$$u(x,t) = u^*(x,t) \quad \text{for a.e. } (x.t) \in Q \tag{159}$$

as soon as the nonlinearity $\beta + \sigma'$ reduces to (7). Indeed, the weak convergence $u^{\varepsilon_k} \rightharpoonup u^*$ in $L^2(0,T;V)$ can be combined with the convergence $(\chi_*^\varepsilon)^3 - \chi_*^\varepsilon \to \chi^3 - \chi$ in $L^2(0,T;H)$. In view of (57), we are thus able to pass to the limit in (55), and obtain that the analogue of (39) (in the case of (7)), holds for χ and u^*. By comparing (39) for u and (39) for u^*, we then deduce that $\langle u(t), 1 \rangle = \langle u^*(t), 1 \rangle$ for a.e. $t \in (0,T)$: combining this with (58), we completely identify u^* and obtain the desired conclusion (159).

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