ASYMPTOTIC ANALYSIS OF THE CAGINALP PHASE-FIELD MODEL FOR TWO VANISHING TIME RELAXATION PARAMETERS

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Abstract

This paper addresses the Caginalp conserved phase-field system, which couples an energy balance equation containing a time relaxation parameter $\varepsilon > 0$ and a source term f, with a Cahn-Allen type dynamics for the order parameter. The analysis is focused on the asymptotic behaviour of the solutions of this parabolic system firstly as $\varepsilon \downarrow 0$, and secondly as both ε and the coefficient $\delta > 0$ of the interfacial energy term in the equation for the order parameter tend to zero. The limit equations are the viscous Cahn-Hilliard equation with source term in the former case and the Cahn-Hilliard equation with source in the latter one. Convergence results are proved, yielding the existence of solutions for both problems, while uniqueness follows from continuous dependence on the data; error estimates are obtained as well. An analogous asymptotic analysis is carried out for the viscous Cahn-Hilliard equation as $\delta \downarrow 0$.

1 Introduction

The present paper is concerned with an initial and homogeneous Neumann boundary value problem for a phase-field system of two parabolic equations governing the evolution of two unknown fields ϑ and χ, namely

(1.1) $$\varepsilon\vartheta_t + \chi_t - \Delta\vartheta = f \quad \text{in } \Omega \times \mathbb{R}^+,$$

(1.2) $$\delta\chi_t - \Delta\chi + \chi^3 - \chi = \vartheta \quad \text{in } \Omega \times \mathbb{R}^+,$$

(where Ω is a bounded, connected domain in $\mathbb{R}^N$, $N = 1, 2, 3$, with smooth boundary $\partial\Omega$, and ε, δ are positive coefficients), supplemented with the boundary conditions

$$\frac{\partial \chi}{\partial n} = \frac{\partial \vartheta}{\partial n} = 0 \quad \text{in } \partial\Omega \times \mathbb{R}^+ \tag{1.3}$$

and the initial conditions

$$\chi(\,\cdot\,, 0) = \chi_0, \quad \vartheta(\,\cdot\,, 0) = \vartheta_0. \tag{1.4}$$

This model is usually investigated in the framework of solid-liquid transitions, so that ϑ is interpreted as the temperature of a physical system (occupying a material region Ω), which may undergo a melting (or conversely a solidification) process; the phase-field χ is an order parameter, yielding the local proportion of the solid and liquid phases, while f represents the heat supply.

In fact, as it is shown in [4, Sec. 4.4], (1.1) follows from the balance law of internal energy, coupled with the Fourier law for the heat flux. On the other hand, (1.2) can be derived from the Ginzburg-Landau theory for the total free-energy functional, assuming in particular for the local free energy density the form of a double-well potential

$$F(\chi) := \frac{(\chi^2 - 1)^2}{4},$$

attaining his two minima correspondingly to the two pure phases, with derivative

$$\phi(\chi) = \chi^3 - \chi. \tag{1.5}$$

A first order approximation around the phase-change temperature yields (1.2); indeed, ϑ turns out to be a relative temperature, scaled in a such a way as to be positive in the liquid phase and negative in the solid one.

This model was first examined by Fix (see [12] and the references therein), and then widely investigated by Caginalp [5], who showed in a subsequent paper [6] that some relevant sharp interface models of phase transitions, i.e. the classical Stefan model and two of its variants, accounting for surface tension effects (see e.g. the book [18, Chap. IV-VIII] for a detailed discussion of these models), arise as limiting cases of the system (1.1), (1.2) as some microscopic parameters go to zero. An analogous formal asymptotic analysis was carried out in [7] for the corresponding *conserved* phase-field system, in which Caginalp eventually obtained the Cahn-Hilliard equation in the limit of zero latent heat and in the particular case of constant temperature.

Later, the paper [16] for the first time investigated the Cahn-Hilliard equation as limit of the *non conserved* system (1.1)-(1.2), thus giving rise to a series of analogous studies: in this framework, we develop an analysis of the phase-field system as the time relaxation parameters ε and δ tend to zero.

To this aim, it is natural to consider the related problem formally obtained by setting $\varepsilon = 0$ in (1.1), (1.2) and substituting the second equation into the first one: hence the phase-field system reduces to

$$\chi_t - \Delta(\delta\chi_t - \Delta\chi + \chi^3 - \chi) = f \quad \text{in } \Omega \times \mathbb{R}^+, \tag{1.6}$$

accordingly supplemented with the Neumann boundary conditions

$$\frac{\partial \chi}{\partial n} = \frac{\partial(\Delta\chi)}{\partial n} = 0 \quad \text{in } \partial\Omega \times \mathbb{R}^+, \tag{1.7}$$

and the initial condition

$$\chi(\,\cdot\,, 0) = \chi_0. \tag{1.8}$$

In the same way, setting $\varepsilon = \delta = 0$, (1.1), (1.2) yield

$$\chi_t - \Delta(-\Delta\chi + \chi^3 - \chi) = f \quad \text{in } \Omega \times \mathbb{R}^+, \tag{1.9}$$

which we couple with (1.7) and (1.8) to define an initial boundary value problem $\mathbf{P}$; for the moment, we will denote by $\mathbf{P}^{\varepsilon\delta}$ the system (1.1)-(1.4), while (1.6)-(1.8) will be referred to as $\mathbf{P}^{\delta}$.

Actually, when $f \equiv 0$, (1.9) is the well-known *Cahn-Hilliard equation*, which describes the evolution of an order parameter χ (e.g., the concentration of a binary alloy) in a system undergoing a phase separation as a result of a quick quenching into a miscibility gap. Similarly, $f \equiv 0$ in (1.6) yields the *viscous Cahn-Hilliard equation*, which was introduced in [13] to account for viscosity effects in the phase separation of, e.g., polymer-polymer systems. It is worthwhile noting that in the phenomena modelled by (1.6) and (1.9) χ is a conserved parameter: therefore, if a source term $f \not\equiv 0$ is to be included in each equation, we have to require it to be spatially homogeneous, i.e.

$$\frac{1}{|\Omega|}\int_\Omega f(x,t)\,dx = 0 \quad \text{for a.e. } t \in (0,T)\ ,$$

in order to retain the conservation property for χ, as we will see later on.

As far as well-posedness results for the problems outlined above are concerned, in [3] the existence of an attractor for the solutions to problem (1.1)-(1.4), (with null f), is demonstrated; actually, in the present paper we refer to the existence result for $\mathbf{P}^{\varepsilon\delta}$ contained in [1, Thm. 2.1], (in fact, [1] more generally tackles a parabolic phase-field system based on the Coleman-Gurtin heat flux law, with an additional memory kernel accounting for memory effects, and with a nonlinear dynamics for χ). An existence and uniqueness theorem for the initial and Neumann homogeneous boundary value problem for the equation (1.6) with source term $f \equiv 0$ was first proved in [11, Thm. 2.3]; among the several papers devoted to the Cahn-Hilliard equation with null source term, we mention the survey [14] and the existence result Thm. 3.1. therein.

As mentioned before, the limiting behaviour of the phase-field model for vanishing time relaxation parameters was first investigated in [16], in which the solution of the system (1.1)-(1.2) with Dirichlet boundary conditions for ϑ and χ was shown to converge as $\varepsilon = \delta \downarrow 0$ to the weak solution of the related boundary value problem for the Cahn-Hilliard equation in arbitrary space dimension. This evidenced a strong link between two basic models in the mathematics of material sciences and motivated a number of papers focused on the asymptotic behaviour of the associated attractors. We quote in particular the paper [11], in which the authors prove that the attractor $\mathcal{A}^{\varepsilon\delta}$ associated to Problem $\mathbf{P}^{\varepsilon\delta}$ is upper-semicontinuous at $(\varepsilon,\delta) = (0,\delta)$, with $\delta > 0$ arbitrary, and at $\varepsilon = \delta = 0$: namely, $\mathcal{A}^{\varepsilon\delta}$ is shown to be asymptotically included in a convenient embedding of the attractor of $\mathbf{P}^\delta$ as $\varepsilon \downarrow 0$, and of the attractor of $\mathbf{P}$ as $\varepsilon = \delta \downarrow 0$; the upper-semicontinuity of the attractor of $\mathbf{P}^\delta$ for vanishing δ is also investigated. Further, the subsequent paper [10] focuses on the lower-semicontinuity of the attractor $\mathcal{A}^{\varepsilon\delta}$ as $\varepsilon \downarrow 0$.

On the other hand, like [16] the present paper is concerned with the asymptotic behaviour of the *finite time* solutions of $\mathbf{P}^{\varepsilon\delta}$ as $\varepsilon \downarrow 0$ or both ε and $\delta \downarrow 0$; apparently, the analysis for $\delta \downarrow 0$ and $\varepsilon > 0$ fixed is much more difficult to handle. As regards the asymptotic behaviour for vanishing ε *and* δ, we point out that, unlike [16], our own analysis is carried out in space dimension $N \le 3$, which enables us to obtain stronger convergence results under weaker assumptions on the regularity of the solutions and on the data, as well as the related error estimates. As expected, the solutions to $\mathbf{P}^{\varepsilon\delta}$ are shown to converge to a solution of (1.6)-(1.8) as ε tends to zero, and of (1.7)-(1.9) as ε, $\delta \downarrow 0$.

In this way, the existence Theorems 2.2 and 2.5 for the limit problems $\mathbf{P}^\delta$ and $\mathbf{P}$ are proved: we note that existence results for such problems have not been yet obtained up to now, as far as the author knows, when the source term is not null. Uniqueness results, (see Thms. 2.1 and 2.4), for $\mathbf{P}^\delta$ and $\mathbf{P}$ are given as well, while Theorems 2.3 and 2.6 provide error estimates of order $O(\varepsilon^{1/2})$ and $O(\varepsilon^{1/2}) + O(\delta)$ for the strong convergences obtained in Theorems 2.2 and 2.5, respectively. Finally, a similar analysis is carried out for the solutions of $\mathbf{P}^\delta$ as $\delta \downarrow 0$.

The outline of the paper is as follows: Section 2 is devoted to the notation, the assumptions and the statement of the results. The asymptotic analyses of $\mathbf{P}^{\varepsilon\delta}$ for vanishing ε and ε, δ are developed in Sections 3 and 4. Once the well-posedness results for $\mathbf{P}^\delta$ and $\mathbf{P}$ are established, the behaviour of $\mathbf{P}^\delta$ for $\delta \downarrow 0$ is examined in Section 5.

2 Statement of the main results

Notation.

As mentioned above, we are interested in the behaviour of the solutions to $\mathbf{P}^{\varepsilon\delta}$, $\mathbf{P}^\delta$ and $\mathbf{P}$ in the "cylinder" $Q = \Omega \times (0, T)$; we set

$$H := L^2(\Omega), \quad V := H^1(\Omega), \quad \text{and } W := \{v \in H^2(\Omega) \,:\, \partial_n v = 0\},$$

∂_n being the outward normal derivative to $\partial\Omega$. We identify H with its dual space H', and recall that the embeddings

$$W \subset V \subset H \subset V' \subset W'$$

are dense and compact. Henceforth, we denote by $(\,\cdot\,,\,\cdot\,)$ the inner product in H, by $\langle\,\cdot\,,\,\cdot\,\rangle$, ($\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle$), the duality pairing between V' and V, (between W' and W), and by $|\cdot|_H$, $\|\cdot\|_V$, $\|\cdot\|_{V'}$ the norms in H (and in H^N, as well), in V and in V'. Besides, we introduce $\mathcal{W}$, $\mathcal{V}$, $\mathcal{H}$, $\mathcal{V}'$ and $\mathcal{W}'$, the subspaces of W, V, H, V' and W' of elements with zero mean value $m(v)$, where

$$m(v) := \frac{1}{|\Omega|}\langle\!\langle v, 1\rangle\!\rangle.$$

Moreover, let us consider the operator $A : V \to V'$ defined by

$$\langle Au, v\rangle := \int_\Omega \nabla u \nabla v \, dx \quad \forall u, v \in V.$$

We remark that $Au \in \mathcal{V}'$ for every $u \in V$ and that $Au = -\Delta u \in H$ whenever $u \in W$; moreover, A can be extended to an operator from H to $\mathcal{W}'$, (which we still denote by A), by means of the variational equality

$$\langle\langle Au, w\rangle\rangle := -\int_\Omega \Delta w u \, dx.$$

Since the kernel of $A : V \to \mathcal{V}'$ consists of all constant functions, the restriction of A to $\mathcal{V}$ turns out to be an isomorphism; thus we can define the inverse operator $\mathcal{N} : \mathcal{V}' \to \mathcal{V}$ by the condition

$$A(\mathcal{N}v) = v \quad \forall v \in \mathcal{V}',$$

namely, $\mathcal{N}v$ is the solution with zero mean value of a generalized Neumann problem with right-hand side v, hence $\mathcal{N}v \in \mathcal{W}$ if $v \in \mathcal{H}$. We point out that

$$\langle Au, \mathcal{N}v\rangle = \langle v, u\rangle \quad \forall u \in V,\ \forall v \in \mathcal{V}', \tag{2.1}$$

$$\langle u, \mathcal{N}v\rangle = \int_\Omega \nabla(\mathcal{N}u)\nabla(\mathcal{N}v)\, dx = \langle v, \mathcal{N}u\rangle \quad \forall u, v \in \mathcal{V}'. \tag{2.2}$$

Henceforth, we will always refer to the following norms in V and V', equivalent to the previous ones on behalf of Poincaré's inequality and denoted by the same symbols:

$$\|u\|_V^2 := \langle Au, u\rangle + (u, m(u)) \quad \forall u \in V \tag{2.3}$$

$$\|v\|_{V'}^2 := \langle v, \mathcal{N}(v - m(v))\rangle + (v, m(v)) \quad \forall v \in V'. \tag{2.4}$$

Statement of the problems and main results.

Within this framework, we can give a rigorous formulation of Problems $\mathbf{P}^{\varepsilon\delta}$, $\mathbf{P}^{\delta}$ and $\mathbf{P}$ in terms of abstract operator equations in the spaces V' and W'.

Problem $\mathbf{P}^{\varepsilon\delta}$. *Given the data $\chi^0, \vartheta^0 \in V$ and $f \in L^2(0,T;H)$, find both χ and ϑ in $H^1(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;W)$ and satisfying*

$$\varepsilon\partial_t\vartheta + \partial_t\chi + A\vartheta = f \quad \textit{in } V',\ \textit{a.e. in } (0,T), \tag{1.1'}$$

$$\delta\partial_t\chi + A\chi + \chi^3 - \chi = \vartheta \quad \textit{in } V',\ \textit{a.e. in } (0,T), \tag{1.2'}$$

with the initial conditions (1.4).

We note that the boundary conditions (1.3) for ϑ and χ are contained in (1.1') and (1.2'), respectively. As mentioned before, [1, Thm. 2.1] ensures that Problem $\mathbf{P}^{\varepsilon\delta}$ admits a unique solution.

Similarly, we can reformulate (1.6)-(1.8).

Problem $\mathbf{P}^\delta$. *Let χ^0 and f fulfil*

$$\chi^0 \in V, \tag{H1}$$

$$f \in L^2(0,T;V'), \quad \frac{1}{|\Omega|}\int_\Omega f(x,t)dx = 0 \quad \text{for a.e. } t \in (0,T) \ . \tag{H2}$$

Then, find $\chi \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)$ satisfying

$$\partial_t \chi + A(\delta \partial_t \chi + A\chi + \chi^3 - \chi) = f \quad \text{in } W', \text{ a.e. in } (0,T), \tag{1.6'}$$

and subject to the initial condition (1.8).

The following continuous dependence result holds for $\mathbf{P}^\delta$.

Theorem 2.1. *Let (f_i, χ_i^0), $i = 1, 2$ be two choices of data for Problem $\mathbf{P}^\delta$ satisfying* (H1) *and* (H2)*; let χ_i, $i = 1, 2$, be the corresponding solutions and put*

$$M := \max_{i=1,2}\{|\chi_i^0|_H^2 + \|f_i\|^2_{L^2(0,T;V')}\}. \tag{2.5}$$

Then there exists a positive constant C_1, depending on M, T, $|\Omega|$ and δ, such that

$$\|\chi_1 - \chi_2\|_{C^0([0,T];H)\cap L^2(0,T;V)} \leq C_1 \left(|\chi_1^0 - \chi_2^0|_H + \|f_1 - f_2\|_{L^2(0,T;V')}\right) . \tag{2.6}$$

It follows that if $\mathbf{P}^\delta$ admits a solution, then it is necessarily unique.

As far as existence of solutions to $\mathbf{P}^\delta$ is concerned, we proceed in the following way. It is possible to approximate the data χ^0 and f of Problem $\mathbf{P}^\delta$ by two sequences $\{\chi_\varepsilon^0\}$ and $\{f_\varepsilon\}$ fulfilling

$$\chi_\varepsilon^0 \in V \ \forall \varepsilon > 0, \qquad \text{and } \chi_\varepsilon^0 \to \chi^0 \text{ in } H \text{ as } \varepsilon \downarrow 0, \tag{H3}$$

$$f_\varepsilon \in L^2(0,T;H), \qquad \text{and } f_\varepsilon \to f \text{ in } L^2(0,T;V') \text{ as } \varepsilon \downarrow 0; \tag{H4}$$

note that (H2) and (H4) entail that $\|m(f_\varepsilon)\|_{L^2(0,T)} \to 0$ as $\varepsilon \downarrow 0$, too. Further, let us consider a sequence $\{\vartheta_\varepsilon^0\} \subset V$. For every $\varepsilon > 0$, let $(\chi_\varepsilon, \vartheta_\varepsilon)$ be the solution of Problem $\mathbf{P}^{\varepsilon\delta}$ corresponding to the fixed interfacial energy coefficient $\delta > 0$, to the source term f_ε and the initial data $(\chi_\varepsilon^0, \vartheta_\varepsilon^0)$: the limiting behaviour of the sequences $\{\chi_\varepsilon\}$ and $\{\vartheta_\varepsilon\}$ is of course tightly related to the issue of "passing to the limit" in (1.1'), (1.2') as the relaxation parameter ε tends to 0. Indeed, we are going to show that, as $\varepsilon \downarrow 0$, $\{\chi_\varepsilon\}$ suitably converges to a function χ which turns out to solve Problem $\mathbf{P}^\delta$: to this aim, we take advantage of an equivalent formulation of (1.6') by means of the auxiliary variable ϑ (cf. (1.1)-(1.2) with $\varepsilon = 0$)

$$\begin{cases} \partial_t \chi + A\vartheta = f \quad \text{in } V', \text{ a.e. in } (0,T), \\ \delta\partial_t \chi + A\chi + \chi^3 - \chi = \vartheta \quad \text{in } V', \text{ a.e. in } (0,T). \end{cases} \tag{2.7}$$

The variable ϑ is often referred to as *chemical potential*; as for Problem $\mathbf{P}^{\varepsilon\delta}$, we note that (2.7) yields homogeneous Neumann conditions for χ and ϑ, which substantially entail, by the definition of the chemical potential, the boundary conditions (1.7). In this setting, the following convergence result holds true, entailing the existence of solutions to $\mathbf{P}^\delta$.

Theorem 2.2. *Assume that* $\{\chi^0_\varepsilon\}$, $\{\vartheta^0_\varepsilon\}$, $\{f_\varepsilon\}$, χ^0, *and* f *fulfil* (H1)-(H4)*; suppose moreover that there exists a constant* $K \geq 0$ *such that*

$$\varepsilon^{1/2}|\vartheta^0_\varepsilon|_H \leq K \quad \forall\, \varepsilon > 0, \tag{H5}$$

$$\|\chi^0_\varepsilon\|_V \leq K \quad \forall\, \varepsilon > 0, \tag{H6}$$

$$\|m(f_\varepsilon)\|_{L^2(0,T)} = O(\varepsilon^{1/2}) \quad \forall\, \varepsilon > 0. \tag{H7}$$

For a fixed $\delta > 0$ *and for every* $\varepsilon > 0$, *let* $(\chi_\varepsilon, \vartheta_\varepsilon)$ *be the solution of* $\mathbf{P}^{\varepsilon\delta}$ *supplemented with* f_ε *and* $(\chi^0_\varepsilon, \vartheta^0_\varepsilon)$*: then there exist* χ *and* ϑ *such that the following strong* ($\rightarrow$)*, weak* ($\rightharpoonup$) *and weak-star* ($\rightharpoonup^*$) *convergences hold as* $\varepsilon \downarrow 0$*:*

$$\chi_\varepsilon \rightharpoonup^* \chi \quad \text{in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \tag{2.8}$$

$$\chi_\varepsilon \rightarrow \chi \quad \text{in } C^0([0,T];H) \cap L^2(0,T;V), \tag{2.9}$$

$$\vartheta_\varepsilon \rightharpoonup \vartheta \quad \text{in } L^2(0,T;V), \tag{2.10}$$

$$\varepsilon\vartheta_\varepsilon \rightarrow 0 \quad \text{in } C^0([0,T];H) \cap L^2(0,T;V), \tag{2.11}$$

$$\varepsilon\vartheta_\varepsilon \rightharpoonup 0 \quad \text{in } H^1(0,T;V'). \tag{2.12}$$

Moreover, χ *and* ϑ *fulfil* (2.7) *and* (1.8)*, hence* χ *is the unique solution of* $\mathbf{P}^\delta$.

We can specify the rate of the strong convergences (2.9), (2.11).

Theorem 2.3. *Under the assumptions of Theorem 2.2, there exists a constant* $S_1 \geq 0$*, depending on* T *and* $|\Omega|$ *only, such that the estimate*

$$\|\chi - \chi_\varepsilon\|_{C^0([0,T];H)\cap L^2(0,T;V)} + \|\varepsilon\vartheta_\varepsilon\|_{C^0([0,T];H)\cap L^2(0,T;V)} \leq$$

$$\leq S_1\left(\varepsilon^{1/2} + |\chi^0 - \chi^0_\varepsilon|_H + \|f - f_\varepsilon\|_{L^2(0,T;V')}\right) \tag{2.13}$$

holds for every $\varepsilon \in (0,1)$.

Let us now rephrase Problem (1.7)-(1.9) in a more rigorous way.

Problem P. *Given* χ^0 *and* f *fulfilling* (H1),(H2)*, find a function* $\chi \in H^1(0,T;V') \cap L^\infty(0,T;V) \cap L^2(0,T;W)$ *such that*

$$\partial_t\chi + A(A\chi + \chi^3 - \chi) = f \quad \text{in } W'\text{, for a.e. } t \in (0,T) \tag{1.9'}$$

and subject to the initial condition (1.8).

We can state a continuous dependence result, entailing uniqueness, for Problem $\mathbf{P}$ as well.

Theorem 2.4. *Let χ_i, $i = 1, 2$, be the solutions of* **P** *corresponding to the initial conditions χ_i^0 and the source terms f_i, $i = 1, 2$ that fulfil* (H1) *and* (H2)*; let*

$$M' := \max_{i=1,2}\{\|\chi_i^0\|_{V'}^2 + \|f_i\|_{L^2(0,T;V')}^2\}.$$

Then there exists a positive constant C_2, depending on M, T and $|\Omega|$, such that

$$\|\chi_1 - \chi_2\|_{C^0([0,T];H)\cap L^2(0,T;V)} \leq C_2 \left(|m(\chi_1^0) - m(\chi_2^0)| + \|f_1 - f_2\|_{L^2(0,T;V')}\right). \tag{2.14}$$

Existence of solutions to Problem **P** is now obtained by "passing to the limit" in $\mathbf{P}^{\varepsilon\delta}$ as both the relaxation parameters ε and δ tend to 0: namely, given the data χ^0 and f of Problem **P**, we consider three sequences $\{\chi_{\varepsilon\delta}^0\}$, $\{\vartheta_{\varepsilon\delta}^0\}$ and $\{f_{\varepsilon\delta}\}$ such that

$$\chi_{\varepsilon\delta}^0, \vartheta_{\varepsilon\delta}^0 \in V\ \forall\, \varepsilon, \delta > 0, \quad \chi_{\varepsilon\delta}^0 \to \chi^0 \quad \text{in } H \text{ as } \varepsilon, \delta \downarrow 0, \tag{H8}$$

$$f_{\varepsilon\delta} \in L^2(0,T;H), \ f_{\varepsilon\delta} \to f \quad \text{in } L^2(0,T;V') \text{ as } \varepsilon, \delta \downarrow 0, \tag{H9}$$

and for every ε, $\delta > 0$ we consider the pair $(\chi_{\varepsilon\delta}, \vartheta_{\varepsilon\delta})$ solving $\mathbf{P}^{\varepsilon\delta}$ with the data $\chi_{\varepsilon\delta}^0$, $\vartheta_{\varepsilon\delta}^0$ and $f_{\varepsilon\delta}$. In order to show that $\chi_{\varepsilon\delta}$ converges to the unique solution of **P**, we preliminarly split (1.9') into a parabolic system featuring the auxiliary variable ϑ (cf. (1.1)-(1.2) with $\varepsilon = \delta = 0$)

$$\begin{cases} \partial_t\chi + A\vartheta = f & \text{in } V', \text{ a.e. in } (0,T), \\ A\chi + \chi^3 - \chi = \vartheta & \text{in } V', \text{ a.e. in } (0,T). \end{cases} \tag{2.15}$$

We are now able to state the following existence result for Problem **P**.

Theorem 2.5. *Assume* (H1),(H2),(H8) *and* (H9) *for χ^0, f, $\{\chi_{\varepsilon\delta}^0\}$, $\{\vartheta_{\varepsilon\delta}^0\}$ and $\{f_{\varepsilon\delta}\}$. Suppose moreover that* (H5)*,* (H6) *and* (H7) *hold for $\{\vartheta_{\varepsilon\delta}^0\}$, $\{\chi_{\varepsilon\delta}^0\}$ and $\{f_{\varepsilon\delta}\}$, respectively.*
Then the sequence of solutions $(\chi_{\varepsilon\delta}, \vartheta_{\varepsilon\delta})$ of $\mathbf{P}^{\varepsilon\delta}$ corresponding to the initial data $(\chi_{\varepsilon\delta}^0, \vartheta_{\varepsilon\delta}^0)$ and the source term $f_{\varepsilon\delta}$ converges as ε, $\delta \downarrow 0$ to the unique pair (χ, ϑ) solving (2.15) *and* (1.8)*; precisely, we have*

$$\chi_{\varepsilon\delta} \rightharpoonup^* \chi \quad \text{in } L^\infty(0,T;V) \cap L^2(0,T;W), \tag{2.16}$$

$$\chi_{\varepsilon\delta} \to \chi \quad \text{in } C^0([0,T];V') \cap L^2(0,T;V), \tag{2.17}$$

$$\vartheta_{\varepsilon\delta} \rightharpoonup \vartheta \quad \text{in } L^2(0,T;V), \tag{2.18}$$

$$\varepsilon\vartheta_{\varepsilon\delta} \to 0 \quad \text{in } C^0([0,T];V') \cap L^2(0,T;V), \tag{2.19}$$

$$\delta\partial_t\chi_{\varepsilon\delta} \to 0 \quad \text{in } L^2(0,T;H), \tag{2.20}$$

$$\varepsilon\partial_t\vartheta_{\varepsilon\delta} + \partial_t\chi_{\varepsilon\delta} \rightharpoonup \partial_t\chi \quad \text{in } L^2(0,T;V'). \tag{2.21}$$

Furthermore, $\chi \in C^0([0,T];H)$ and it is the unique solution of **P**.

It is possible to derive some error estimates in this case as well.

Theorem 2.6. *Let the structural assumptions of Theorem 2.5 hold. Then there exists a constant $S_2 \geq 0$, depending on T and $|\Omega|$ only, such that*

$$\|\chi - \chi_{\varepsilon\delta}\|_{C^0([0,T];V')\cap L^2(0,T;V)}$$

$$\leq S_2 \left(\|\chi^0 - \chi^0_{\varepsilon\delta}\|^2_{V'} + \delta^{1/2}|\chi^0 - \chi^0_{\varepsilon\delta}|_H + \|f - f_{\varepsilon\delta}\|_{L^2(0,T;V')} + \varepsilon^{1/2} + \delta\right) \tag{2.22}$$

holds for every ε, $\delta \in (0,1)$; the same estimate holds for $\|\varepsilon\vartheta_{\varepsilon\delta}\|_{C^0([0,T];V')\cap L^2(0,T;V)}$, too.

In the end, we turn to examining the asymptotic behaviour of the solutions of $\mathbf{P}^\delta$ as $\delta \downarrow 0$ and prove the convergence results below.

Theorem 2.7. *Let χ^0, $\{\chi^0_\delta\} \subset V$, f, $\{f_\delta\} \subset L^2(0,T;V')$ fulfil*

$$f_\delta(t), f(t) \in \mathcal{V}' \quad \text{a.e. in } (0,T) \quad \forall\, \delta > 0, \tag{H10}$$

$$f_\delta \to f \quad \text{in } L^2(0,T;V') \text{ as } \delta \downarrow 0, \tag{H11}$$

$$\chi^0_\delta \to \chi^0 \quad \text{in } H \text{ as } \delta \downarrow 0, \tag{H12}$$

$$\|\chi^0_\delta\|_V \leq K \tag{H13}$$

for a constant K independent of δ; let $\{\chi_\delta\}$ (χ) be the solution of $\mathbf{P}^\delta$ (of $\mathbf{P}$) with source term f_δ (f) and initial condition χ^0_δ (χ^0). Then we have as $\delta \downarrow 0$

$$\chi_\delta \rightharpoonup^* \chi \quad \text{in } L^\infty(0,T;V) \cap H^1(0,T;V'), \tag{2.23}$$

$$\chi_\delta \to \chi \quad \text{in } C^0([0,T];H) \cap L^2(0,T;W), \tag{2.24}$$

$$\delta\partial_t\chi_\delta \to 0 \quad \text{in } L^2(0,T;H). \tag{2.25}$$

Theorem 2.8. *Under the assumptions of Theorem 2.7, there exists a constant $S_3 \geq 0$, depending on T and $|\Omega|$, such that*

$$\|\chi - \chi_\delta\|_{C^0(0,T;V')\cap L^2(0,T;V)} \leq S_3 \left(\|\chi^0 - \chi^0_\delta\|_{V'} + \|f - f_\delta\|_{L^2(0,T;V')} + \delta\right) \tag{2.26}$$

and

$$\|\chi - \chi_\delta\|_{C^0([0,T];H)\cap L^2(0,T;W)} \leq S_3 \left(|\chi^0 - \chi^0_\delta|_H + \|f - f_\delta\|_{L^2(0,T;V')} + \delta^{1/2}\right) \tag{2.27}$$

for every $\delta \in (0,1)$.

Throughout the next sections, devoted to the proofs of the results stated above, we will widely use the well-known Young inequality

$$\forall p \in (1,\infty) \quad \forall \mu > 0 \;\exists\, c_\mu : \quad xy \leq \mu x^p + c_\mu y^{p'} \quad \forall x, y \geq 0, \quad \frac{1}{p} + \frac{1}{p'} = 1, \tag{2.28}$$

and the two following inequalities for the functional ϕ (cf. (1.5))

$$\forall \alpha \in [0,1) \quad \exists\, C_\alpha \geq 0 : \quad \phi(r)r \geq \alpha r^4 - C_\alpha \quad \forall r \in \mathbb{R}, \tag{2.29}$$

$$\forall \eta > 0 \quad \exists\, C_\eta > 0 \quad \phi(r) \leq \eta r^4 + c_\eta \quad \forall r \in \mathbb{R}. \tag{2.30}$$

3 Asymptotic behaviour of $\mathbf{P}^{\varepsilon\delta}$ as $\varepsilon \downarrow 0$

For every pair $(\chi_\varepsilon, \vartheta_\varepsilon)$ as in Theorem 2.2, it may be convenient to introduce the "enthalpy" function

$$u_\varepsilon := \varepsilon\vartheta_\varepsilon + \chi_\varepsilon \tag{3.1}$$

Testing (1.1') by $v = 1$ yields

$$\tfrac{d}{dt} m(u_\varepsilon(t)) = m(f_\varepsilon(t)) \quad \text{a.e. in } (0,T),\ \forall\, \varepsilon > 0, \tag{3.2}$$

whence

$$E_\varepsilon(t) := m(u_\varepsilon(t)) = m(\varepsilon\vartheta_\varepsilon^0 + \chi_\varepsilon^0) + \int_0^t m(f_\varepsilon(s))ds, \tag{3.3}$$

so that, on account of (H3)-(H5), there exists a constant $S \geq 0$ such that

$$|E_\varepsilon(t)| \leq S \quad \forall t \in [0,T], \tag{3.4}$$

for every $\varepsilon \in (0,1)$, say. Arguing in the same way, we see that if χ solves Problem $\mathbf{P}^\delta$ with the initial condition $\chi^0 \in V$ and the source f as in (H2), then its mean value is conserved, i.e.

$$m(\chi(t)) = m_0 := m(\chi^0) \quad \forall t \in [0,T]. \tag{3.5}$$

Throughout the proofs of Theorems 2.1-2.8, we adopt the general convention of denoting by the same symbol C several constants depending only on the quantities specified by the statement of each theorem, and possibly on the initial data, pointing out the few occurring exceptions.

Continuous dependence on the data for Problem $\mathbf{P}^\delta$*.*

The proof of Theorem 2.1 relies on the techniques developed in [8, Lemma 3.1], so that here we briefly outline the argument for the reader's convenience, referring to [8] for more detailed computations.

Remark 3.1. Preliminarly, we point out that if $\chi \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)$ is a solution of $\mathbf{P}^\delta$ with data χ^0 and f, then there exists a positive constant C such that

$$\|\chi\|_{L^4(Q)}^4 \leq C\left(1 + m_0^4 + |\chi^0|_H^2 + \|f\|_{L^2(0,T;V')}^2\right) \tag{3.6}$$

Indeed, testing (1.6') by $\mathcal{N}(\chi(t) - m_0) \in \mathcal{W}$ for a.e. $t \in (0,T)$ (in view of (3.5)), and integrating on $(0,t)$, on account of (2.28), (2.29) and of (2.3), (2.4) we obtain

$$\frac{1}{2}\|\chi(t) - m_0\|_{V'}^2 + \frac{\delta}{2}|\chi(t) - m_0|_H^2 + \int_0^t \|\chi(s) - m_0\|_V^2 ds + \alpha \int_0^t \left(\int_\Omega \chi^4(x,s)dx\right) ds$$

$$\leq C + \frac{1}{2}\|\chi^0 - m_0\|_{V'}^2 + \frac{\delta}{2}|\chi^0 - m_0|_H^2 + \int_0^t \left(\chi^3(s) - \chi(s), m_0\right) ds$$

$$+\frac{1}{2}\|f\|^2_{L^2(0,T;V')} + \frac{1}{2}\int_0^t \|\chi(s) - m_0\|^2_{V'} ds \tag{3.7}$$

for a.e. $t \in (0,T)$ and for some constant $0 < \alpha < 1$. By virtue of (2.28), we have

$$\left|\int_0^t \left(\chi^3(s), m_0\right) ds\right| \le \frac{\alpha}{2}\int_0^t \left(\int_\Omega \chi^4(x,s)dx\right) ds + Cm_0^4,$$

while

$$\left|\int_0^t \left(\chi(s), m_0\right) ds\right| \le \left|\int_0^t \langle m_0, \chi(s) - m_0\rangle ds\right| + \int_0^t |m_0|^2_H ds \le \frac{1}{2}\|\chi - m_0\|^2_{L^2(0,t;V)} + Cm_0^2.$$

Thus (3.7) reads

$$\frac{1}{2}\|\chi(t) - m_0\|^2_{V'} + \frac{1}{2}\|\chi - m_0\|^2_{L^2(0,t;V)} + \frac{\alpha}{2}\int_0^t \left(\int_\Omega \chi^4(x,s)dx\right) ds$$

$$\le C\left(1 + m_0^4 + |\chi^0|^2_H\right) + \frac{1}{2}\|f\|^2_{L^2(0,T;V')} + \frac{1}{2}\|\chi - m_0\|^2_{L^2(0,t;V')}.$$

By applying Gronwall's Lemma (see, e.g., [2, Lemma A.4]) to $\|\chi(t) - m_0\|^2_{V'}$, we conclude (3.6).

Proof of Theorem 2.1. We set

$$\overline{\chi} := \chi_1 - \chi_2, \quad \overline{\chi}^0 := \chi_1^0 - \chi_2^0, \quad \overline{f} := f_1 - f_2,$$

and note that the function $m(\overline{\chi})$ takes the constant value $\overline{m_0} := m(\chi_1^0) - m(\chi_2^0)$; in view of (1.6'), $\overline{\chi}$ fulfils

$$\partial_t\overline{\chi} + A(\delta\partial_t\overline{\chi} + A\overline{\chi} + \chi_1^3 - \chi_2^3 - \overline{\chi}) = \overline{f} \quad \text{in } W', \text{ a.e. in } (0,T).$$

Testing by $\mathcal{N}(\overline{\chi}(t) - \overline{m_0})$ for a.e. $t \in (0,T)$ and integrating in time, on behalf of the monotonicity of $y \mapsto y^3$ we easily get

$$\frac{1}{2}\|\overline{\chi}(t) - \overline{m_0}\|^2_{V'} + \frac{\delta}{2}|\overline{\chi}(t) - \overline{m_0}|^2_H + \int_0^t \|\overline{\chi}(s) - \overline{m_0}\|^2_V ds$$

$$\le C\left(|\overline{\chi}^0 - \overline{m_0}|^2_H + \|\overline{f}\|^2_{L^2(0,T;V')} + \|\overline{\chi} - \overline{m_0}\|^2_{L^2(0,t;V')}\right)$$

$$+ \int_0^t \left(\chi_1^3(s) - \chi_2^3(s), \overline{m_0}\right) ds + \int_0^t \left(\overline{\chi}(s), \overline{\chi}(s) - \overline{m_0}\right) ds$$

Estimating the two final terms separately by

$$\left|\overline{m_0}\int_0^t \left(\int_\Omega \overline{\chi}(\chi_1^2 + \chi_2^2 + \chi_1\chi_2)dx\right) ds\right|$$

$$\le |\overline{m_0}|\int_0^t |\langle \chi_1^2 + \chi_2^2 + \chi_1\chi_2, \overline{\chi} - \overline{m_0}\rangle| ds + |\overline{m_0}|^2 \int_0^t |\langle \chi_1^2 + \chi_2^2 + \chi_1\chi_2, 1\rangle|\, ds$$

$$\leq \frac{1}{8}\|\overline{\chi}-\overline{m_0}\|^2_{L^2(0,t;V)} + C|\overline{m_0}|^2\left(1+\|\chi_1\|^4_{L^4(Q)}+\|\chi_2\|^4_{L^4(Q)}\right), \tag{3.8}$$

and by

$$\left|\int_0^t \langle \overline{\chi}, \overline{\chi}-\overline{m_0}\rangle\, ds\right| \leq C\left(\|\overline{\chi}-\overline{m_0}\|^2_{L^2(0,t;V')}+|\overline{m_0}|^2\right)+\frac{1}{8}\|\overline{\chi}-\overline{m_0}\|^2_{L^2(0,t;V)}, \tag{3.9}$$

respectively, invoking (3.6) and Gronwall's Lemma for $\|\overline{\chi}(t)-\overline{m_0}\|^2_{V'}$, we infer (2.6). □

Remark 3.2. Following [8, Rem. 3.6], we remark that, thanks to the continuous dependence estimate (2.6), it is possible to relax the requirement on the initial condition χ^0 of Problem $\mathbf{P}^\delta$ to be in V, extending the notion of solution to $\mathbf{P}^\delta$ to the case in which $\chi^0 \in H$. Indeed, let us consider a sequence $\{\chi^0_\eta\} \subset V$ approximating χ^0 in H: then the sequence $\{\chi_\eta\}$ of the corresponding solutions of $\mathbf{P}^\delta$ fulfils the Cauchy condition in $C^0([0,T];H)\cap L^2(0,T;V)$ thanks to (2.6). Taking the limit of $\{\chi_\eta\}$, we obtain therefore a *generalized* solution $\chi \in C^0([0,T];H)\cap L^2(0,T;V)$ of Problem $\mathbf{P}^\delta$, corresponding to the initial condition $\chi^0 \in H$.

Proof of Theorem 2.2. The first part of the proof consists of obtaining a priori estimates for the sequences of solutions $\{\chi_\varepsilon\}$ and $\{\vartheta_\varepsilon\}$, in order to extract subsequences converging to the unique solution of Problem $\mathbf{P}^\delta$ in suitable topologies. To this aim, we test equations (1.1'), (1.2') by appropriate functions.

First a priori estimate. Following [11, Lemma 3.1], and choosing $\mathcal{N}(u_\varepsilon - E_\varepsilon) \in \mathcal{W}$ as test function for (1.1'), we get

$$\langle \partial_t(u_\varepsilon(t)-E_\varepsilon(t)), \mathcal{N}(u_\varepsilon(t)-E_\varepsilon(t))\rangle + \langle A\vartheta_\varepsilon(t), \mathcal{N}(u_\varepsilon(t)-E_\varepsilon(t))\rangle$$
$$= \langle f_\varepsilon(t), \mathcal{N}(u_\varepsilon(t)-E_\varepsilon(t))\rangle \quad \text{for a.e. } t\in(0,T).$$

Integrating in time, in view of (2.1), (2.4) and (3.1) we derive

$$\frac{1}{2}\|u_\varepsilon(t)-E_\varepsilon(t)\|^2_{V'} + \varepsilon\int_0^t |\vartheta_\varepsilon(s)|^2_H\, ds + \int_0^t (\vartheta_\varepsilon(s), \chi_\varepsilon(s)-E_\varepsilon(s))\, ds$$

$$\leq \frac{1}{2}\left(\|\varepsilon\vartheta^0_\varepsilon + \chi^0_\varepsilon - E_\varepsilon(0)\|^2_{V'} + \|f_\varepsilon\|^2_{L^2(0,T;V')} + \|u_\varepsilon - E_\varepsilon\|^2_{L^2(0,t;V')}\right) \tag{3.10}$$

Substituting (1.2') for ϑ_ε, noting that

$$\langle A\chi_\varepsilon(t), \chi_\varepsilon(t)-E_\varepsilon(t)\rangle = \|\chi_\varepsilon(t)-m(\chi_\varepsilon)(t)\|^2_V \quad \text{for a.e. } t\in(0,T)$$

by (2.3), and that

$$(\partial_t\chi_\varepsilon(t), \chi_\varepsilon(t)-E_\varepsilon(t)) = (\partial_t(\chi_\varepsilon - E_\varepsilon)(t), \chi_\varepsilon(t)-E_\varepsilon(t)) + (m(f_\varepsilon)(t), \chi_\varepsilon(t)-E_\varepsilon(t))$$

by (3.2), we can rewrite the third term in the left-hand side of (3.10) as

$$\frac{\delta}{2}|\chi_\varepsilon(t)-E_\varepsilon(t)|^2_H - \frac{\delta}{2}|\chi^0_\varepsilon - E_\varepsilon(0)|^2_H + \delta\int_0^t (m(f_\varepsilon)(s), \chi_\varepsilon(s)-E_\varepsilon(s))\, ds$$

$$(3.11)\qquad + \int_0^t \|\chi_\varepsilon(s) - m(\chi_\varepsilon)(s)\|_V^2\, ds + \int_0^t (\chi_\varepsilon^3(s) - \chi_\varepsilon(s), \chi_\varepsilon(s) - E_\varepsilon(s))ds.$$

Note that

$$\delta \int_0^t (m(f_\varepsilon)(s), \chi_\varepsilon(s) - E_\varepsilon(s))ds$$

$$(3.12)\qquad \leq \delta^2 \int_0^t |m(f_\varepsilon)(s)|_H^2\, ds + \frac{1}{2}\int_0^t |\chi_\varepsilon(s) - m(\chi_\varepsilon)(s)|_H^2\, ds + \frac{1}{2}\int_0^t |\varepsilon m(\vartheta_\varepsilon)(s)|_H^2\, ds.$$

Handling the last term in (3.11) by means of (2.29) (with the choice $\alpha = 1/2$), and of (2.30) (with $\eta = 1/(4S)$ and S as in (3.4)), yields

$$(3.13)\qquad \int_0^t (\chi_\varepsilon^3(s) - \chi_\varepsilon(s), \chi_\varepsilon(s))ds - \int_0^t (\chi_\varepsilon^3(s) - \chi_\varepsilon(s), E_\varepsilon(s))ds \geq \frac{1}{4}\int_0^t \left(\int_\Omega \chi_\varepsilon^4(x,s)dx\right) ds - C.$$

Collecting (3.10)-(3.13), we obtain

$$\frac{1}{2}\|u_\varepsilon(t) - E_\varepsilon(t)\|_{V'}^2 + \varepsilon \int_0^t |\vartheta_\varepsilon(s)|_H^2\, ds + \frac{\delta}{2}|\chi_\varepsilon(t) - E_\varepsilon(t)|_H^2 + \|\chi_\varepsilon - m(\chi_\varepsilon)\|_{L^2(0,t;V)}^2$$

$$+\frac{1}{4}\int_0^t \left(\int_\Omega \chi_\varepsilon^4(x,s)dx\right) ds \leq C\left(1 + \varepsilon^2\|\vartheta_\varepsilon^0\|_{V'}^2 + |\chi_\varepsilon^0|_H^2 + \|m(f_\varepsilon)\|_{L^2(0,T)}^2\right)$$

$$(3.14)\qquad +\frac{1}{2}\left(\|f_\varepsilon\|_{L^2(0,T;V')}^2 + \|u_\varepsilon - E_\varepsilon\|_{L^2(0,t;V')}^2 + \|\chi_\varepsilon - m(\chi_\varepsilon)\|_{L^2(0,t;V)}^2 + \varepsilon^2\int_0^t |\vartheta_\varepsilon(s)|_H^2 ds\right).$$

Taking into account (H3), (H4), applying Gronwall's Lemma, and recalling (3.4) as well, we deduce from (3.14) the estimate

$$(3.15)\qquad \|u_\varepsilon\|_{L^\infty(0,T;V')} + \|\varepsilon^{1/2}\vartheta_\varepsilon\|_{L^2(0,T;H)} + \|\chi_\varepsilon\|_{L^4(Q)\cap L^2(0,T;V)} \leq C$$

for some positive constant C independent of ε. We observe that, as $\delta > 0$ is a fixed constant by hypothesis, (3.14) also implies that $\{\chi_\varepsilon\}$ is bounded in $L^\infty(0,T;H)$.

Second a priori estimate. Following the regularization argument devised in, e.g., [9] (see [9, Lemma 2.1] in particular), let us consider, for any fixed $\varepsilon > 0$ and $t \in (0,T)$, the V-valued (unique) solution $\chi_\varepsilon^\mu(t)$ of the elliptic variational problem

$$(3.16)\qquad (\chi_\varepsilon^\mu(t), \zeta) + \mu^2\Big((\chi_\varepsilon^\mu(t), \zeta) + \langle A\chi_\varepsilon^\mu(t), \zeta\rangle\Big) = (\chi_\varepsilon(t), \zeta) \quad \forall \zeta \in V,$$

where $\mu > 0$. As shown in [9], since $\chi_\varepsilon \in H^1(0,T;H)$, (3.16) defines a function $\chi_\varepsilon^\mu \in H^1(0,T;V)$ for every $\mu > 0$, and for a.e. $t \in (0,T)$ $\partial_t\chi_\varepsilon^\mu(t)$ solves the corresponding variational equation with $\partial_t\chi_\varepsilon(t)$ on the right-hand side.

Let us test (1.1') by $\vartheta_\varepsilon(t)$ and (1.2') by $\partial_t\chi_\varepsilon^\mu(t)$: adding the two equations and integrating on $(0,t)$, we obtain

$$\delta\int_0^t (\partial_t\chi_\varepsilon(s), \partial_t\chi_\varepsilon^\mu(s))\, ds + \int_0^t \langle A\chi_\varepsilon(s), \partial_t\chi_\varepsilon^\mu(s)\rangle\, ds + \int_0^t (\chi_\varepsilon^3(s) - \chi_\varepsilon(s), \partial_t\chi_\varepsilon^\mu(s))\, ds$$

$$+\int_0^t (\partial_t\chi_\varepsilon(s) - \partial_t\chi_\varepsilon^\mu(s), \vartheta_\varepsilon(s))\, ds + \frac{\varepsilon}{2}|\vartheta_\varepsilon(t)|_H^2 + \int_0^t \langle A\vartheta_\varepsilon(s), \vartheta_\varepsilon(s)\rangle\, ds$$

$$= \frac{\varepsilon}{2}|\vartheta_\varepsilon^0|_H^2 + \int_0^t (f_\varepsilon(s), \vartheta_\varepsilon(s))\, ds \tag{3.17}$$

for a.e. $t \in (0,T)$. Now, let us take the limit as $\mu \downarrow 0$ of (3.17), $\varepsilon > 0$ being fixed: we observe that

$$\lim_{\mu\downarrow 0}\Big|\int_0^t (\partial_t\chi_\varepsilon(s), \partial_t\chi_\varepsilon^\mu(s) - \partial_t\chi_\varepsilon(s))\, ds\Big| \le \lim_{\mu\downarrow 0}\int_0^t |\partial_t\chi_\varepsilon(s)|_H |\partial_t\chi_\varepsilon^\mu(s) - \partial_t\chi_\varepsilon(s)|_H\, ds = 0$$

since, by [9, Prop. 6.1], $\partial_t\chi_\varepsilon^\mu(t) \to \partial_t\chi_\varepsilon(t)$ as $\mu \downarrow 0$ and $|\partial_t\chi_\varepsilon^\mu(t)|_H \le |\partial_t\chi_\varepsilon(t)|$ for every $\mu > 0$, for a.e. $t \in (0,T)$. We likewise conclude that

$$\lim_{\mu\downarrow 0}\int_0^t (\chi_\varepsilon^3(s) - \chi_\varepsilon(s), \partial_t\chi_\varepsilon^\mu(s))\, ds = \int_0^t (\chi_\varepsilon^3(s) - \chi_\varepsilon(s), \partial_t\chi_\varepsilon(s))\, ds,$$

$$\lim_{\mu\downarrow 0}\int_0^t (\partial_t\chi_\varepsilon(s) - \partial_t\chi_\varepsilon^\mu(s), \vartheta_\varepsilon(s))\, ds = 0,$$

while, on behalf of [9, Prop. 6.3], we have

$$\lim_{\mu\downarrow 0}\int_0^t \langle A\chi_\varepsilon(s), \partial_t\chi_\varepsilon^\mu(s)\rangle\, ds = \frac{1}{2}\langle A\chi_\varepsilon(t), \chi_\varepsilon(t)\rangle - \frac{1}{2}\langle A\chi_\varepsilon^0, \chi_\varepsilon^0\rangle.$$

As a result, (3.17) yields

$$\int_0^t |\delta^{\frac12}\partial_t\chi_\varepsilon(s)|_H^2\, ds + \frac{\varepsilon}{2}|\vartheta_\varepsilon(t)|_H^2 + \int_\Omega \left(\frac{\chi_\varepsilon^4(x,t)}{4} - \frac{\chi_\varepsilon^2(x,t)}{2}\right) dx$$

$$+\frac{1}{2}\|\chi_\varepsilon(t) - m(\chi_\varepsilon(t))\|_V^2 + \int_0^t \|\vartheta_\varepsilon(s) - m(\vartheta_\varepsilon)(s)\|_V^2\, ds$$

$$\le \frac{\varepsilon}{2}|\vartheta_\varepsilon^0|_H^2 + \int_\Omega \Big(\frac{(\chi_\varepsilon^0)^4}{4} - \frac{(\chi_\varepsilon^0)^2}{2}\Big)\, dx + \frac{1}{2}|\nabla\chi_\varepsilon^0|_H^2 + \int_0^t (f_\varepsilon(s), \vartheta_\varepsilon(s))\, ds \tag{3.18}$$

for a.e. $t \in (0,T)$. We estimate the last summand in (3.18) by

$$\frac{1}{2}\left(\|f_\varepsilon\|_{L^2(0,T;V')}^2 + \int_0^t \|\vartheta_\varepsilon(s) - m(\vartheta_\varepsilon)(s)\|_V^2\, ds\right) + C\|m(f_\varepsilon)\|_{L^2(0,T)}\|m(\vartheta_\varepsilon)\|_{L^2(0,T)}$$

and observe that the latter term is bounded, owing to (H7) and (3.15). In the end, we deduce that

$$\|\delta^{1/2}\partial_t\chi_\varepsilon\|_{L^2(0,T;H)} + \|\varepsilon^{1/2}\vartheta_\varepsilon\|_{L^\infty(0,T;H)} \le C; \tag{3.19}$$

owing to (3.15), we can estimate $\{m(\chi_\varepsilon)\}$ in $L^\infty(0,T)$ as well, so that (3.18) entails the upper bound

$$\|\chi_\varepsilon\|_{L^\infty(0,T;V)} + \|\vartheta_\varepsilon - m(\vartheta_\varepsilon)\|_{L^2(0,T;V)} \le C. \tag{3.20}$$

Therefore, in view of the continuous embedding $V \subset L^6(\Omega)$, we have that

$$\{\chi_\varepsilon^3\} \subset L^\infty(0,T;H) \quad \text{is bounded.} \tag{3.21}$$

Finally, since δ is a fixed parameter, (3.19) yields the estimate

$$\|\partial_t \chi_\varepsilon\|_{L^2(0,T;H)} \leq C \tag{3.22}$$

as well. By (3.19) and (3.20), a comparison in (1.2') entails that $\{\vartheta_\varepsilon\} \subset L^2(0,T;V')$ is bounded; clearly, the same conclusion holds for $\{m(\vartheta_\varepsilon)\} \subset L^2(0,T)$, whence

$$\{\vartheta_\varepsilon\} \subset L^2(0,T;V) \quad \text{is bounded,} \tag{3.23}$$

thanks to (3.20). Eventually, by comparison in (1.1'), on account of (H4) as well, we deduce

$$\|\partial_t u_\varepsilon\|_{L^2(0,T;V')} = \|\varepsilon \partial_t \vartheta_\varepsilon + \partial_t \chi_\varepsilon\|_{L^2(0,T;V')} \leq C. \tag{3.24}$$

Third a priori estimate. Finally, we test (1.2') by $A\chi_\varepsilon(t)$ (the aforementioned [9, Prop. 6.1-6.3] again ensure that the argument is correct) and integrate on $(0,T)$, obtaining, thanks to (2.28),

$$\int_0^T |A\chi_\varepsilon(t)|_H^2 dt \leq \|\nabla \vartheta_\varepsilon\|_{L^2(0,T;H)} \|\nabla \chi_\varepsilon\|_{L^2(0,T;H)}$$
$$+2\|\delta \partial_t \chi_\varepsilon\|^2_{L^2(0,T;H)} + 2\|\chi_\varepsilon^3 - \chi_\varepsilon\|^2_{L^2(0,T;H)} + \frac{1}{2}\|A\chi_\varepsilon\|^2_{L^2(0,T;H)}.$$

Thus, in view of (3.21) and (3.22), we infer that there exists a constant $C \geq 0$ such that

$$\|\chi_\varepsilon\|_{L^2(0,T;W)} \leq C \quad \forall \varepsilon > 0. \tag{3.25}$$

Passage to the limit. Let us first remark that (3.20), (3.22) and (3.25) entail that $\{\chi_\varepsilon\}$ is relatively compact in both the strong topology of $L^2(0,T;V)$ and of $C^0([0,T];H)$, by Lions-Aubin's Theorem (see [15, Thm. 5]) and by [15, Cor. 4], respectively; further, on account of (3.23) as well, well-known weak and weak-star compactness results ensure that there exist two subsequences (which we still denote by the same symbol $\{\chi_\varepsilon\}$, $\{\vartheta_\varepsilon\}$ for the sake of simplicity) and two functions $\chi \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)$ and $\vartheta \in L^2(0,T;V)$ for which (2.8), (2.10), and (2.9) hold true. Moreover, taking into account (3.19) and (3.23), we have $\varepsilon\vartheta_\varepsilon \to 0$ in $C^0([0,T];H)$ and in $L^2(0,T;V)$, whence

$$\varepsilon\vartheta_\varepsilon + \chi_\varepsilon \to \chi \quad \text{in } C^0([0,T];H) \cap L^2(0,T;V) \tag{3.26}$$

and, by (3.24),

$$\varepsilon\vartheta_\varepsilon + \chi_\varepsilon \rightharpoonup \chi \quad \text{in } H^1(0,T;V'). \tag{3.27}$$

Note that, in principle, (3.21) only implies that there exists a function $\gamma \in L^\infty(0,T;H)$ such that

$$\chi_\varepsilon^3 \rightharpoonup^* \gamma \quad \text{in } L^\infty(0,T;H). \tag{3.28}$$

Nevertheless, (2.9) ensures that $\chi_\varepsilon^3 \to \chi^3$ a.e. in Q for a subsequence, hence by Egorov's theorem one necessarily has $\gamma = \chi^3$ a.e. in Q.

In conclusion, thanks to (2.8), (2.9), (2.10), (3.27) and (3.28), it is possible to pass to the limit in (1.1'), (1.2') and verify that (2.7) holds for the pair (χ, ϑ); χ fulfils the initial condition (1.8) as well, on behalf of (2.9) and (H3). Since the solution of Problem $\mathbf{P}^\delta$ is unique, the limit pair (χ, ϑ) does *not* depend on the subsequence extracted, so that the convergences listed above hold for the whole families $\{\chi_\varepsilon\}$, $\{\vartheta_\varepsilon\}$ as $\varepsilon \downarrow 0$. □

Error estimates.

Proof of Theorem 2.3. Setting

$$\widetilde{\chi}_\varepsilon := \chi - \chi_\varepsilon, \quad \widetilde{\vartheta}_\varepsilon := \vartheta - \vartheta_\varepsilon, \quad \widetilde{E}_\varepsilon := m(\chi) - E_\varepsilon = m(\widetilde{\chi}_\varepsilon) - \varepsilon m(\vartheta_\varepsilon), \quad \widetilde{f}_\varepsilon := f - f_\varepsilon,$$

we observe that, by (3.2) and (3.5),

$$\tfrac{d}{dt}\widetilde{E}_\varepsilon(t) = -m(f_\varepsilon(t)) \quad \text{a.e. in } (0,T), \tag{3.29}$$

and, on behalf of (3.3), (H5), and (H7),

$$\left|\widetilde{E}_\varepsilon(t)\right| \le C\left(\|\chi^0 - \chi^0_\varepsilon\|_{V'} + \varepsilon^{1/2}\right) \tag{3.30}$$

for some constant C independent of $\varepsilon > 0$ and $t \in (0,T)$. Further, it follows from (3.19) and (3.23), respectively, that

$$\|m(\widetilde{\chi}_\varepsilon) - \widetilde{E}_\varepsilon\|_{L^\infty(0,T)} = O(\varepsilon^{1/2}) \quad \text{and } \|m(\widetilde{\chi}_\varepsilon) - \widetilde{E}_\varepsilon\|_{L^2(0,T)} = O(\varepsilon). \tag{3.31}$$

Subtracting (1.1') from the first equation of (2.7), testing by $\mathcal{N}(\widetilde{\chi}_\varepsilon(t) - \varepsilon\vartheta_\varepsilon(t) - \widetilde{E}_\varepsilon(t)) \in \mathcal{W}$ for a.e. $t \in (0,T)$, integrating in time and arguing as for (3.10), we obtain

$$\frac{1}{2}\|\widetilde{\chi}_\varepsilon(t) - \varepsilon\vartheta_\varepsilon(t) - \widetilde{E}_\varepsilon(t)\|^2_{V'} + \int_0^t \left(\widetilde{\vartheta}_\varepsilon(s), \widetilde{\chi}_\varepsilon(s) - \varepsilon\vartheta_\varepsilon(s) - \widetilde{E}_\varepsilon(s)\right) ds$$

$$= \frac{1}{2}\|\chi^0 - \chi^0_\varepsilon - \varepsilon\vartheta^0_\varepsilon - \widetilde{E}_\varepsilon(0)\|^2_{V'} + \int_0^t \langle \widetilde{f}_\varepsilon(s), \mathcal{N}(\widetilde{\chi}_\varepsilon(s) - \varepsilon\vartheta_\varepsilon(s) - \widetilde{E}_\varepsilon(s))\, ds$$

$$\le C\left(\|\chi^0 - \chi^0_\varepsilon\|^2_{V'} + \varepsilon\right) + \frac{1}{2}\left(\|\widetilde{f}_\varepsilon\|^2_{L^2(0,T;V')} + \|\widetilde{\chi}_\varepsilon - \varepsilon\vartheta_\varepsilon - \widetilde{E}_\varepsilon\|^2_{L^2(0,t;V')}\right), \tag{3.32}$$

the last inequality following from (H5), (3.30) and the Young inequality (2.28). After replacing $\widetilde{\vartheta}_\varepsilon$ by $\delta\partial_t\widetilde{\chi}_\varepsilon + A\widetilde{\chi}_\varepsilon + \chi^3 - \chi^3_\varepsilon - \widetilde{\chi}_\varepsilon$ on behalf of (1.2') and (2.7), rearranging some terms and taking into account (3.29) too, the last term in the left-hand side of (3.32) is equal to

$$\frac{\delta}{2}|\widetilde{\chi}_\varepsilon(t) - \widetilde{E}_\varepsilon(t)|^2_H - \frac{\delta}{2}|\chi^0 - \chi^0_\varepsilon - \widetilde{E}_\varepsilon(0)|^2_H - \delta\int_0^t \left(m(f_\varepsilon(s)), \widetilde{\chi}_\varepsilon(s) - \widetilde{E}_\varepsilon(s)\right) ds$$

$$+\|\widetilde{\chi}_\varepsilon - m(\widetilde{\chi}_\varepsilon)\|^2_{L^2(0,t;V)} - \varepsilon\int_0^t \left(\widetilde{\vartheta}_\varepsilon(s), \vartheta_\varepsilon(s)\right) ds + \int_0^t \left(\chi^3(s) - \chi^3_\varepsilon(s), \widetilde{\chi}_\varepsilon(s)\right) ds$$

$$-\int_0^t \left(\chi^3(s) - \chi^3_\varepsilon(s), \widetilde{E}_\varepsilon(s)\right) ds - \int_0^t \left(\widetilde{\chi}_\varepsilon(s), \widetilde{\chi}_\varepsilon(s) - \widetilde{E}_\varepsilon(s)\right) ds. \tag{3.33}$$

By Hölder's inequality and (H7), we have

$$\delta\int_0^t \left(m(f_\varepsilon(s)), \widetilde{\chi}_\varepsilon(s) - \widetilde{E}_\varepsilon(s)\right) ds \le C\delta^2\varepsilon + \frac{1}{8}\|\widetilde{\chi}_\varepsilon - \widetilde{E}_\varepsilon\|^2_{L^2(0,t;V)}; \tag{3.34}$$

aiming at comparing $\|\widetilde{\chi}_\varepsilon - \widetilde{E}_\varepsilon\|^2_{L^2(0,t;V)}$ with $\|\widetilde{\chi}_\varepsilon - m(\widetilde{\chi}_\varepsilon)\|^2_{L^2(0,t;V)}$ in (3.33), it is convenient to notice that, by (3.31),

$$\|\widetilde{\chi}_\varepsilon - \widetilde{E}_\varepsilon\|^2_{L^2(0,t;V)} \leq 2\|\widetilde{\chi}_\varepsilon - m(\widetilde{\chi}_\varepsilon)\|^2_{L^2(0,t;V)} + O(\varepsilon^2). \tag{3.35}$$

On the other hand, thanks to (3.23) we have

$$\left|\varepsilon \int_0^t (\widetilde{\vartheta}_\varepsilon(s), \vartheta_\varepsilon(s))\, ds\right| \leq \varepsilon \|\widetilde{\vartheta}_\varepsilon\|_{L^2(0,T;H)} \|\vartheta_\varepsilon\|_{L^2(0,T;H)} \leq C\varepsilon, \tag{3.36}$$

while

$$\int_0^t \big(\chi^3(s) - \chi^3_\varepsilon(s), \widetilde{\chi}_\varepsilon(s)\big)\, ds \geq 0 \tag{3.37}$$

by the monotonicity property of $y \mapsto y^3$. We can handle the last two summands in (3.33) in the same way as in (3.8) and (3.9) and subsequently apply (3.35) and the estimate (3.15) to $\|\chi_\varepsilon\|_{L^4(Q)}$. Collecting (3.34)-(3.37) and recalling (3.30) again, we infer from (3.32) and (3.33)

$$\frac{1}{2}\|\widetilde{\chi}_\varepsilon(t) - \varepsilon\vartheta_\varepsilon(t) - \widetilde{E}_\varepsilon(t)\|^2_{V'} \leq C(|\chi^0 - \chi^0_\varepsilon|^2_H + \varepsilon + \|\widetilde{f}_\varepsilon\|^2_{L^2(0,T;V')} + \|\widetilde{\chi}_\varepsilon - \varepsilon\vartheta_\varepsilon - \widetilde{E}_\varepsilon\|^2_{L^2(0,t;V')}).$$

Invoking Gronwall's Lemma, we deduce an estimate of order $O(\varepsilon^{1/2})$ for $\|\widetilde{\chi}_\varepsilon - \varepsilon\vartheta_\varepsilon - \widetilde{E}_\varepsilon\|_{C^0([0,T];V')}$. A comparison in (3.32), as well as (3.30), (3.31), and (3.33), allow us to conclude the error estimate (2.13) for $\{\widetilde{\chi}_\varepsilon\} \subset C^0([0,T];H) \cap L^2(0,T;V)$; the estimates for $\{\varepsilon\vartheta_\varepsilon\}$ follows immediately from (3.19) and (3.23). □

4 Asymptotic behaviour of $\mathbf{P}^{\varepsilon\delta}$ as $\varepsilon, \delta \downarrow 0$

Proof of Theorem 2.4. Arguing as in the previous section, we observe that the mean value of any solution χ to Problem **P** with a source term f fulfilling (H2) is conserved, i.e. (3.5) holds.
Performing the same estimate as in Remark 3.1, we obtain the analogue of (3.6) for χ

$$\|\chi\|^4_{L^4(Q)} \leq C\left(1 + m_0^4 + \|\chi^0\|^2_{V'} + \|f\|^2_{L^2(0,T;V')}\right), \tag{4.1}$$

noting that the only difference with respect to (3.6) is that the term $|\chi^0|^2_H$ is no longer necessary to estimate $\|\chi\|^4_{L^4(Q)}$, due to the lack of the viscosity term $A(\delta\partial_t\chi)$ in (1.9').
Working out the same argument as in the proof of Theorem 2.1, it is easy to derive (2.14) from (4.1). □

Remark 4.1. The extension outlined in Remark 3.2 can be considered for Problem **P** as well: indeed, by the continuous dependence estimate (2.14), the requirement on the initial condition χ^0 to be in V can be relaxed to $\chi^0 \in V'$ with an analogous argument, thus obtaining a *generalized* solution $\chi \in C^0([0,T];H) \cap L^2(0,T;V)$ to Problem **P**.

Proof of Theorem 2.5. Proceeding as for (3.1), we introduce the enthalpy $u_{\varepsilon\delta}$ associated to the solutions $\chi_{\varepsilon\delta}$, $\vartheta_{\varepsilon\delta}$ of $\mathbf{P}^{\varepsilon\delta}$; let us point out that (3.2)-(3.4) still hold true for its mean value $E_{\varepsilon\delta}$.

The estimates (3.14), (3.18) and (3.25) obtained throughout the proof of Theorem 2.2, the assumptions (H5)-(H9) and similar comparison arguments allow to infer the folllowing upper bounds

$$\|u_{\varepsilon\delta}\|_{H^1(0,T;V')} + \|\chi_{\varepsilon\delta}\|_{L^\infty(0,T;V)\cap L^2(0,T;W)} + \|\vartheta_{\varepsilon\delta}\|_{L^2(0,T;V)}$$

$$+\|\delta^{1/2}\partial_t\chi_{\varepsilon\delta}\|_{L^2(0,T;H)} + \|\varepsilon^{1/2}\vartheta_{\varepsilon\delta}\|_{L^\infty(0,T;H)} \leq C, \tag{4.2}$$

for a positive constant C independent of ε and δ. Hence, (2.20) follows; we also deduce from (4.2) that there exists a pair (χ, ϑ) such that (2.16), (2.18), and (2.21) hold true as ε, $\delta \downarrow 0$ along some subsequence. Further, on behalf of [15, Cor. 4], $\{\varepsilon\vartheta_{\varepsilon\delta} + \chi_{\varepsilon\delta}\}$ is relatively compact in $C^0([0,T];V')$; since $\varepsilon\vartheta_{\varepsilon\delta} \to 0$ in $C^0([0,T];H)$, we have

$$\chi_{\varepsilon\delta} \to \chi \quad \text{in } C^0([0,T];V') \text{ as } \varepsilon,\, \delta \downarrow 0. \tag{4.3}$$

To conclude (2.17), it suffices to remark that by well-known interpolation results

$$\|\chi_{\varepsilon\delta} - \chi\|^2_{L^2(0,T;V)} \leq \|\chi_{\varepsilon\delta} - \chi\|^2_{L^2(0,T;V')}\|\chi_{\varepsilon\delta} - \chi\|^{1/2}_{L^2(0,T;V)}\|\chi_{\varepsilon\delta} - \chi\|_{L^2(0,T;W)},$$

so that, in view of (4.3) and (4.2), $\chi_{\varepsilon\delta} \to \chi$ strongly in $L^2(0,T;V)$ as ε, $\delta \downarrow 0$. Arguing as in the proof of Theorem 2.2, we have

$$\chi^3_{\varepsilon\delta} \overset{*}{\rightharpoonup} \chi^3 \quad \text{in } L^2(0,T;H). \tag{4.4}$$

Finally, since $\chi \in L^\infty(0,T;V) \cap C^0([0,T];V')$, on behalf of [17, Lemma III.1.4] we conclude that χ is weakly continuous in $[0,T]$ with values in V; the inclusion $V \subset H$ being compact, we deduce the further regularity $\chi \in C^0([0,T];H)$.

In view of the convergences proved so far, we deduce that the limit pair (χ, ϑ) fulfils (2.15) and that (1.8) holds for χ, which turns out to be the unique solution of Problem $\mathbf{P}$; as ϑ is unique as well, we retrieve (2.16)-(2.21) for the whole sequences $\{\chi_{\varepsilon\delta}\}$ and $\{\vartheta_{\varepsilon\delta}\}$. □

Error estimates.

Proof of Theorem 2.6. Let us label $\widetilde{\chi}_{\varepsilon\delta} := \chi - \chi_{\varepsilon\delta}$, $\widetilde{\vartheta}_{\varepsilon\delta} := \vartheta - \vartheta_{\varepsilon\delta}$, $\widetilde{f_{\varepsilon\delta}} := f - f_{\varepsilon\delta}$ and $\widetilde{E}_{\varepsilon\delta} := E - E_{\varepsilon\delta}$, observing that (3.29)-(3.31) hold for $\widetilde{E}_{\varepsilon\delta}$ and $m(\widetilde{\chi}_{\varepsilon\delta})$ for every ε and $\delta > 0$, as well.

We take the difference between the first equation of (2.15) and (1.1'), we test it by $\mathcal{N}(\widetilde{\chi}_{\varepsilon\delta}) - \varepsilon\vartheta_{\varepsilon\delta} - \widetilde{E}_{\varepsilon\delta})$ and integrate on $(0,t)$ for a.e. $t \in (0,T)$, thus obtaining the analog of (3.32)

$$\frac{1}{2}\|\widetilde{\chi}_{\varepsilon\delta}(t) - \varepsilon\vartheta_{\varepsilon\delta}(t) - \widetilde{E}_{\varepsilon\delta}(t)\|^2_{V'} + \int_0^t \Big(\widetilde{\vartheta}_{\varepsilon\delta}(s), \widetilde{\chi}_{\varepsilon\delta}(s) - \varepsilon\vartheta_{\varepsilon\delta}(s) - \widetilde{E}_{\varepsilon\delta}(s)\Big)\, ds$$

$$\leq C\left(\|\chi^0-\chi^0_{\varepsilon\delta}\|^2_{V'}+\varepsilon\right)+\frac{1}{2}\left(\|\widetilde{f_{\varepsilon\delta}}\|^2_{L^2(0,T;V')}+\|\widetilde{\chi}_{\varepsilon\delta}-\varepsilon\vartheta_{\varepsilon\delta}-\widetilde{E}_{\varepsilon\delta}\|^2_{L^2(0,t;V')}\right). \tag{4.5}$$

On account of (1.2') and (2.15), we have now $\widetilde{\vartheta}_{\varepsilon\delta}=-\delta\partial_t\chi_{\varepsilon\delta}+A\widetilde{\chi}_{\varepsilon\delta}+\chi^3-\chi^3_{\varepsilon\delta}-\widetilde{\chi}_{\varepsilon\delta}$: rearranging some terms, we see that the second summand in the left-hand side of (4.5) is equal to

$$\delta\int_0^t(\partial_t\widetilde{\chi}_{\varepsilon\delta}(s)-\partial_t\widetilde{E}_{\varepsilon\delta}(s),\widetilde{\chi}_{\varepsilon\delta}(s)-\widetilde{E}_{\varepsilon\delta}(s))\,ds$$

$$-\delta\int_0^t\left(\partial_t\chi(s)-m(f_{\varepsilon\delta})(s),\widetilde{\chi}_{\varepsilon\delta}(s)-\varepsilon\vartheta_{\varepsilon\delta}(s)-\widetilde{E}_{\varepsilon\delta}(s)\right)ds-\delta\int_0^t(\partial_t\chi(s)-m(f_{\varepsilon\delta})(s),\varepsilon\vartheta_{\varepsilon\delta}(s))\,ds$$

$$+\int_0^t\langle A\widetilde{\chi}_{\varepsilon\delta}(s),\widetilde{\chi}_{\varepsilon\delta}(s)-m(\widetilde{\chi}_{\varepsilon\delta})(s)\rangle\,ds+\int_0^t(\chi^3(s)-\chi^3_{\varepsilon\delta}(s),\widetilde{\chi}_{\varepsilon\delta}(s))\,ds$$

$$-\int_0^t(\chi^3(s)-\chi^3_{\varepsilon\delta}(s),\widetilde{E}_{\varepsilon\delta}(s))\,ds-\int_0^t(\widetilde{\chi}_{\varepsilon\delta}(s),\widetilde{\chi}_{\varepsilon\delta}(s)-\widetilde{E}_{\varepsilon\delta}(s))\,ds-\int_0^t(\widetilde{\vartheta}_{\varepsilon\delta}(s),\varepsilon\vartheta_{\varepsilon\delta}(s))\,ds \tag{4.6}$$

We estimate the second term in (4.6) by

$$C\left(\delta^2\|\partial_t\chi\|^2_{L^2(0,T;V')}+\delta^2\varepsilon+\varepsilon^2\right)+\frac{1}{4}\|\widetilde{\chi}_{\varepsilon\delta}-m(\widetilde{\chi}_{\varepsilon\delta})\|^2_{L^2(0,t;V)}, \tag{4.7}$$

on behalf of (H7) and (3.31), and the third summand by

$$C(\varepsilon\delta\|\partial_t\chi\|_{L^2(0,T;H)}\|\vartheta_{\varepsilon\delta}\|_{L^2(0,T;H)}+\varepsilon^2\|m(f_{\varepsilon\delta})\|^2_{L^2(0,T)}+\delta^2\|\vartheta_{\varepsilon\delta}\|^2_{L^2(0,T;H)})\leq C(\varepsilon\delta+\varepsilon+\delta^2), \tag{4.8}$$

by (H7) and the estimate (4.2). By means of (3.8), (3.9) and (3.35), also remarking that

$$\|\widetilde{\chi}_{\varepsilon\delta}-\widetilde{E}_{\varepsilon\delta}\|^2_{L^2(0,t;V')}\leq 2\|\widetilde{\chi}_{\varepsilon\delta}-\varepsilon\vartheta_{\varepsilon\delta}-\widetilde{E}_{\varepsilon\delta}\|^2_{L^2(0,t;V')}+2\varepsilon^2\|\vartheta_{\varepsilon\delta}\|^2_{L^2(0,T;V')},$$

we have the following upper bound for the three remaining terms

$$\varepsilon\|\vartheta_{\varepsilon\delta}\|_{L^2(0,T;H)}\|\widetilde{\vartheta}_{\varepsilon\delta}\|_{L^2(0,T;H)}+C\|\widetilde{E}_{\varepsilon\delta}\|^2_{L^\infty(0,T)}\left(1+\|\chi\|^4_{L^4(Q)}+\|\chi_{\varepsilon\delta}\|^4_{L^4(Q)}+\varepsilon^2\right)$$

$$+C\|\widetilde{\chi}_{\varepsilon\delta}-\widetilde{E}_{\varepsilon\delta}\|^2_{L^2(0,t;V')}+\frac{1}{2}\|\widetilde{\chi}_{\varepsilon\delta}-m(\widetilde{\chi}_{\varepsilon\delta})\|^2_{L^2(0,t;V)}\leq$$

$$\leq C(\varepsilon+\varepsilon^3+\varepsilon^2\|\vartheta_{\varepsilon\delta}\|^2_{L^2(0,T;V')}+\|\widetilde{\chi}_{\varepsilon\delta}-\varepsilon\vartheta_{\varepsilon\delta}-\widetilde{E}_{\varepsilon\delta}\|^2_{L^2(0,t;V')})+\frac{1}{2}\|\widetilde{\chi}_{\varepsilon\delta}-m(\widetilde{\chi}_{\varepsilon\delta})\|^2_{L^2(0,t;V)}. \tag{4.9}$$

Now collecting (4.7)-(4.9) and applying (3.30) to estimate $|\widetilde{E}_{\varepsilon\delta}(t)|$, by the aforementioned monotonicity argument we derive from (4.5)

$$\frac{1}{2}\|\widetilde{\chi}_{\varepsilon\delta}(t)-\varepsilon\vartheta_{\varepsilon\delta}(t)-\widetilde{E}_{\varepsilon\delta}(t)\|^2_{V'}+\frac{\delta}{2}|\widetilde{\chi}_{\varepsilon\delta}(t)-\widetilde{E}_{\varepsilon\delta}(t)|^2_H+\int_0^t\|\widetilde{\chi}_{\varepsilon\delta}(s)-m(\widetilde{\chi}_{\varepsilon\delta})(s)\|^2_V ds\leq$$

$$\leq C\left(\|\chi^0-\chi^0_{\varepsilon\delta}\|^2_V+\varepsilon+\delta|\chi^0-\chi^0_{\varepsilon\delta}|^2_H+\varepsilon\delta+\|\widetilde{f_{\varepsilon\delta}}\|^2_{L^2(0,T;V')}+\delta^2\right)$$

$$
(4.10) \qquad +\frac{3}{4}\|\widetilde{\chi}_{\varepsilon\delta} - m(\widetilde{\chi}_{\varepsilon\delta})\|^2_{L^2(0,t;V)} + C\|\widetilde{\chi}_{\varepsilon\delta} - \varepsilon\vartheta_{\varepsilon\delta} - \widetilde{E}_{\varepsilon\delta}\|^2_{L^2(0,t;V')}.
$$

Then, applying [2, Lemma A.4] to $\widetilde{\chi}_{\varepsilon\delta} - \varepsilon\vartheta_{\varepsilon\delta} - \widetilde{E}_{\varepsilon\delta}$, we conclude

$$
\|\widetilde{\chi}_{\varepsilon\delta} - \varepsilon\vartheta_{\varepsilon\delta} - \widetilde{E}_{\varepsilon\delta}\|^2_{C^0(0,T;V')} \leq
$$

$$
\leq C\left(\|\chi^0 - \chi^0_{\varepsilon\delta}\|^2_{V'} + \varepsilon + \delta|\chi^0 - \chi^0_{\varepsilon\delta}|^2_H + \varepsilon\delta + \|\widetilde{f_{\varepsilon\delta}}\|^2_{L^2(0,T;V')} + \delta^2\right).
$$

On account of (4.10), (2.22) easily follows. □

5 Asymptotic behaviour of $\mathbf{P}^\delta$ as $\delta \downarrow 0$

Proof of Theorem 2.7. As in the proofs of Theorems 2.2 and 2.5, at first we establish some a priori estimates for the sequence of solutions $\{\chi_\delta\}$ of $\mathbf{P}^\delta$.

First a priori estimate. Testing (1.6') by $N(\partial_t\chi_\delta(t)) \in \mathcal{W}$ for a.e. $t \in (0,T)$ (we recall that

$$
m(\partial_t\chi_\delta)(t) = 0 \quad \text{and } m(\chi_\delta)(t) \equiv m(\chi^0_\delta) \text{ for a.e. } t \in (0,T),
$$

as pointed out in (3.5)), integrating in time and reminding (2.3), (2.4), and (2.29) we get

$$
\int_0^t \|\partial_t\chi_\delta(s)\|^2_{V'}ds + \delta\int_0^t |\partial_t\chi_\delta(s)|^2_H ds + \frac{1}{2}\|\chi_\delta(t) - m(\chi^0_\delta)\|^2_V + \alpha\int_\Omega \chi^4_\delta(x,t)dx
$$

$$
(5.1) \qquad \leq C + \frac{1}{2}\|\chi^0_\delta - m(\chi^0_\delta)\|^2_V + \frac{1}{4}\|\chi^0_\delta\|^4_{L^4(\Omega)} + \frac{1}{2}\|f_\delta\|^2_{L^2(0,T;V')} + \frac{1}{2}\int_0^t \|\partial_t\chi_\delta\|^2_{V'}ds.
$$

for some constant $0 < \alpha < 1$. Owing to (H11) and (H13), (5.1) entails the uniform bound

$$
(5.2) \qquad \|\partial_t\chi_\delta\|_{L^2(0,T;V')} + \|\delta^{\frac{1}{2}}\partial_t\chi_\delta\|_{L^2(0,T;H)} + \|\chi_\delta\|_{L^\infty(0,T;V)} \leq C.
$$

Then, by [15, Cor. 4] we can conclude that $\{\chi_\delta\}$ is relatively strongly compact in $C^0([0,T];H)$.

Second a priori estimate. Testing (1.6') by $\chi_\delta(t)$ and integrating on $(0,t)$ for a.e. $t \in (0,T)$, we obtain

$$
\int_0^T |A\chi_\delta(s)|^2_H\, ds \leq C|\chi^0_\delta|^2_H
$$

$$
+C\left(\delta^2\|\partial_t\chi_\delta\|^2_{L^2(0,T;H)} + \|\chi^3_\delta - \chi_\delta\|^2_{L^2(0,T;H)} + \|f_\delta\|^2_{L^2(0,T;V')} + \|\chi_\delta\|^2_{L^2(0,T;V)}\right).
$$

By well-known elliptic regularity results, this entails that

$$
(5.3) \qquad \{\chi_\delta\} \subset L^2(0,T;W) \quad \text{is bounded,}
$$

in view of (5.2); further, thanks to Lions-Aubin's theorem [15, Thm. 5] we infer from (5.2), (5.3) that

$$
(5.4) \qquad \{\chi_\delta\} \quad \text{is relatively strongly compact in } L^2(0,T;V).
$$

We can now verify that $\{\chi_\delta\}$ fulfils the Cauchy condition in $L^2(0,T;W)$, at least along some subsequence. Indeed, let us write (1.6') for two distinct solutions χ_μ and χ_ν, corresponding to the parameters δ_μ, δ_ν and to the source terms f_μ, f_ν; subtracting the equations, testing by $\chi_\mu(t)-\chi_\nu(t)$ for a.e $t\in(0,T)$ and integrating on $(0,T)$ we deduce

$$\int_0^T |A\left(\chi_\mu(t)-\chi_\nu(t)\right)|_H^2\,dt \le C\left(|\chi_\mu^0-\chi_\nu^0|_H^2+\|f_\mu-f_\nu\|^2_{L^2(0,T;V')}\right)$$

$$+C\left(\delta_\mu\|\delta_\mu^{\frac12}\partial_t\chi_\mu\|^2_{L^2(0,T;H)}+\delta_\nu\|\delta_\nu^{\frac12}\partial_t\chi_\nu\|^2_{L^2(0,T;H)}\right)$$

$$+C\left(\|\chi_\mu^3-\chi_\nu^3\|^2_{L^2(0,T;H)}+\|\chi_\mu-\chi_\nu\|^2_{L^2(0,T;V)}\right) \tag{5.5}$$

To conclude that $\{A\chi_\delta\}\subset L^2(0,T;H)$ is a Cauchy sequence, it suffices to show that all the terms in (5.5) tend to 0 as δ_μ, $\delta_\nu\downarrow 0$: in view of (H11), (H12), (5.2) and (5.4), this reduces to checking that

$$\|\chi_\mu^3-\chi_\nu^3\|^2_{L^2(0,T;H)}\le\frac92\int_0^T\left(\int_\Omega|\chi_\mu-\chi_\nu|^2|\chi_\mu^4+\chi_\nu^4|\,dx\right)dt$$

$$\le\frac92\int_0^T\left(\|\chi_\mu^4\|_{L^{3/2}(\Omega)}+\|\chi_\nu^4\|_{L^{3/2}(\Omega)}\right)\left(\|(\chi_\mu-\chi_\nu)^2\|_{L^3(\Omega)}\right)dt$$

$$\le C\int_0^T\|\chi_\mu-\chi_\nu\|^{1/3}_{L^6(\Omega)}\,dt\le C\|\chi_\mu-\chi_\nu\|^2_{L^2(0,T;V)}\to 0\quad\text{as }\delta_\mu,\ \delta_\nu\downarrow 0, \tag{5.6}$$

in view of (5.4) and since (5.2) entails that $\{\chi_\delta^4\}\subset L^\infty(0,T;L^{3/2}(\Omega))$ is bounded.

The argument developed so far provides a limit function $\chi\in L^\infty(0,T;V)\cap L^2(0,T;W)\cap C^0([0,T];H)$ and some subsequence (which we still denote by $\{\chi_\delta\}$), for which (2.23)-(2.25), in addition to $\chi_\delta^3\rightharpoonup^*\chi_\delta^3$ in $L^\infty(0,T;H)$, hold true. Thus, it is possible to pass to the limit in (1.6') and conclude that χ fulfils (1.9') and the initial condition (1.8); by uniqueness, besides (2.25), the convergences (2.23), (2.24) as well hold for the whole sequence $\{\chi_\delta\}$. □

Proof of Theorem 2.8. Preliminarily, we are going to find an error estimate for $\widetilde\chi_\delta:=\chi-\chi_\delta$ in $L^2(0,T;V)$. To this aim, let us take the difference between (1.9') and (1.6') (denoting the term $f-f_\delta$ by $\tilde f_\delta$), and test it by $N(\widetilde\chi_\delta(t)-m(\widetilde\chi_\delta))$; an integration on $(0,t)$ yields, by monotonicity,

$$\frac12\|\widetilde\chi_\delta(t)-m(\widetilde\chi_\delta)\|^2_{V'}+\int_0^t\|\widetilde\chi_\delta(s)-m(\widetilde\chi_\delta)(s)\|^2_V ds\le C\|\chi^0-\chi^0_\delta\|^2_{V'}+\frac12\|\tilde f_\delta\|^2_{L^2(0,T;V')}$$

$$+\frac12\|\widetilde\chi_\delta-m(\widetilde\chi_\delta)\|^2_{L^2(0,t;V')}+2\delta^2\|\partial_t\chi_\delta\|^2_{L^2(0,T;V')}+\frac14\|\widetilde\chi_\delta-m(\widetilde\chi_\delta)\|^2_{L^2(0,T;V)}$$

$$+\int_0^t\left(\chi^3(s)-\chi_\delta^3(s),m(\widetilde\chi_\delta)\right)ds+\int_0^t\left(\widetilde\chi_\delta(s),\widetilde\chi_\delta(s)-m(\widetilde\chi_\delta)\right)ds \tag{5.7}$$

taking into account that $|m(\widetilde{\chi}_\delta)(t)|^2 \leq C\|\chi^0 - \chi^0_\delta\|^2_{V'}$ for a.e. $t \in (0,T)$. Estimating the two summands above with the help of (3.8) and (3.9), recalling (5.2), (which implies in particular that $\chi_\delta \subset L^4(Q)$ is bounded), and applying Gronwall's Lemma to $\|\widetilde{\chi}_\delta(t) - m(\widetilde{\chi}_\delta)\|^2_{V'}$, we deduce

$$\|\chi - \chi_\delta\|^2_{C^0(0,T;V')\cap L^2(0,T;V)} \leq K_1 \left(\|\chi^0 - \chi^0_\delta\|^2_{V'} + \|f - f_\delta\|^2_{L^2(0,T;V')} + \delta^2 \right). \tag{5.8}$$

It is now easy to provide an error estimate for $\|\widetilde{\chi}_\delta\|_{C^0([0,T];H)}$ and $\|\widetilde{\chi}_\delta\|_{L^2(0,T;W)}$: indeed, choosing $\widetilde{\chi}_\delta(t)$ as test function and integrating in time for all $t \in (0,T)$, we get

$$|\widetilde{\chi}_\delta(t)|^2_H + \int_0^t |A\widetilde{\chi}_\delta(s)|^2_H ds \leq C \left(|\chi^0 - \chi^0_\delta|^2_H + \|f - \tilde{f}_\delta\|^2_{L^2(0,T;V')} \right)$$

$$+C \left(\|\widetilde{\chi}_\delta\|^2_{L^2(0,T;V)} + \delta \|\delta^{\frac{1}{2}} \partial_t \chi_\delta\|^2_{L^2(0,T;H)} + \|\chi^3 - \chi^3_\delta\|^2_{L^2(0,T;H)} + \|\widetilde{\chi}_\delta\|^2_{L^2(0,T;H)} \right). \tag{5.9}$$

The point here is that we already dispose of the error estimate (5.8), so that, estimating the third term in the line above as in (5.6), on account of (5.2) as well, we conclude that there exists a constant $K_2 \geq 0$ fulfilling

$$\|\chi - \chi_\delta\|^2_{C^0([0,T];H)\cap L^2(0,T;W)} \leq K_2 \left(|\chi^0 - \chi^0_\delta|^2_H + \|f - f_\delta\|^2_{L^2(0,T;V')} + \delta \right).$$

□

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