Compactness results for evolution equations

Riccarda Rossi *

Abstract

La compattezza nello spazio $L^p(0,T;B)$ è stata studiata, fra gli altri, da Aubin [1] e da Simon [13], che hanno ottenuto una caratterizzazione dei sottoinsiemi relativamente compatti in L^p in termini di un opportuno funzionale integrale. In questa nota il problema è stato esaminato alla luce della teoria delle misure di Young per la semicontinuità inferiore, e sono state fornite nuove condizioni necessarie e sufficienti per la compattezza; inoltre, è stata affrontata la questione, connessa, della compattezza rispetto alla convergenza in misura, per la quale è stato ottenuto un criterio generale. Nell'ultima parte sono state proposte applicazioni al problema di Stefan.

1 Compactness in $L^p(0,T;B)$

Let us consider a bounded family $\mathcal{U}$ of functions in $L^p(0,T;B)$, where B is a Banach space and $1 \le p < \infty$. When B is of *finite* dimension, then the celebrated Theorem of Riesz-Fréchet-Kolmogorov (see e.g. [3, Thm. IV.26]) says that $\mathcal{U}$ is totally bounded in $L^p(0,T;B)$ if and only if

$$\lim_{h\downarrow 0} \int_0^{T-h} \|u(t+h)-u(t)\|_B^p dt = 0 \quad \text{uniformly for } u \in \mathcal{U}, \tag{1}$$

i.e. there exists a uniform modulus of continuity $\omega : [0,+\infty[\to [0,+\infty[$ with $\omega(0)=0$ and $\lim_{h\to 0}\omega(h)=0$ such that

$$\int_0^{T-h} \|u(t+h)-u(t)\|_B^p dt \le \omega(h) \quad \forall h \in (0,T),\ u \in \mathcal{U}. \tag{2}$$

Usually, in many evolution problems, (2) is provided by exhibiting a uniform estimate in $W^{1,p}(0,T;B)$ for the functions in $\mathcal{U}$, (in that case, $\omega(h) =$

*Dipartimento di Matematica "Felice Casorati", Università di Pavia, Via Ferrata, 1 - 27100 Pavia. E-mail: riccarda@dimat.unipv.it

$h \sup_{u\in\mathcal{U}} \|u'\|_{L^p(0,T;B)})$, or even in a Sobolev-Besov space of fractional order (see e.g. [9], [4]).

If the dimension of B is not finite, then conditions (1), (2) are no longer sufficient to ensure the relative compactness and some extra condition should be imposed on $\mathcal{U}$: roughly speaking, the idea is that the values of the functions $u \in \mathcal{U}$ should belong, in a suitable integral sense, to some compact set of B. A general sufficient condition, who played a crucial role in the so called "compactness method" for nonlinear evolution problems as showed in [8], was given by J.P. Aubin [1] in the case of reflexive spaces, by assuming that there exists another Banach space $A \subset B$ such that

$$\text{the inclusion } A \subset B \text{ is compact, } \mathcal{U} \text{ is bounded in } L^p(0,T;A). \tag{3}$$

It should be remarked that, once (3) holds, in (2) the norm of B could be replaced by the weaker norm of any Banach space C in which B is continuosly contained (see Remarks 1.5 and 2.2).

J.Simon [13] has provided a complete characterization of the compact sets in $L^p(0,T;B)$, showing that it is necessary and sufficient for $\mathcal{U}$ to satisfy (1) and

$$\left\{ \int_0^t u(s)ds \ : u \in \mathcal{U} \right\} \quad \text{is relatively compact in } B \quad \forall t \in (0,T). \tag{4}$$

It is easy to see that (3) is stronger than (4); on the other hand, (3) seems easier to handle in many applications, where it follows directly from *a priori* estimates involving the values of $u \in \mathcal{U}$ instead of its time integrals.

So it is natural to wonder whether (4) can be replaced by another (necessary and sufficient) condition closer to (3). In order to understand in what direction we can generalize (3), let us rephrase it in a slightly different form: we introduce the functional $\mathcal{F}_{p,A} : B \to [0,+\infty]$

$$\mathcal{F}_{p,A}(v) := \begin{cases} \|v\|_A^p & \text{if } v \in A, \\ +\infty & \text{if } v \in B \setminus A \end{cases} \tag{5}$$

It is easy to see that $\mathcal{F}_{p,A}$ is lower semicontinuous and that its sublevels

$$\{v \in B \ : \mathcal{F}_{p,A}(v) \le c\} \quad c \in [0,+\infty)$$

are *compact* in B. (3) is then equivalent to

$$\sup_{u\in\mathcal{U}} \int_0^T \mathcal{F}_{p,A}(u(t))dt < +\infty \tag{6}$$

A natural idea is to replace $\mathcal{F}_{p,A}$ by a general *normal integrand* $\mathcal{F} : (0,T) \times B \to [0,+\infty]$ with compact sublevels: more precisely, denoting by $\mathcal{L}$ and $\mathcal{B}$ the σ-algebras of the Lebesgue-measurable subsets of $(0,T)$ and of the Borel subsets of B respectively, we recall that $\mathcal{F}$ is a *normal coercive integrand* if

$$\mathcal{F} : (0,T) \times B \to [0,+\infty] \quad \text{is } \mathcal{L} \otimes \mathcal{B}\text{-measurable}, \tag{7}$$

$$\text{the map } v \mapsto \mathcal{F}(t,v) \text{ is l.s.c. for a.e. } t \in (0,T), \tag{8}$$

$$\text{for a.e. } t \in (0,T) \quad \{v \in B \ : \mathcal{F}(t,v) \le c\} \text{ is compact } \forall c \in \mathbf{R}. \tag{9}$$

These conditions were introduced by E.J.Balder [2] in developing a Young measure framework for studying lower semicontinuity in optimal control problems; following [2, §2], we say that $\mathcal{U}$ is *tight* (w.r.t. $\mathcal{F}$) if

$$\sup_{u \in \mathcal{U}} \int_0^T \mathcal{F}(t,u(t))dt < +\infty \tag{10}$$

Our first result (the proof of this and of the following theorems will be given in a forthcoming paper) concerns a characterization of $L^p(0,T;B)$ where (4) can be replaced by (10):

Theorem 1.1 *A bounded family $\mathcal{U} \subset L^p(0,T;B)$ is relatively compact if and only if (1) holds and $\mathcal{U}$ satisfies (10) for a normal coercive integrand $\mathcal{F}$.*

The proof of Theorem 1.1 relies on the fundamental lower semicontinuity result of Young measure theory [2, Th.1] (as a matter of fact, the tightness condition (10) for the functions $u \in \mathcal{U}$ is equivalent to the tightness notion of probability theory for the set of Young measures associated to $\mathcal{U}$): the idea is to reduce the problem of compactness in $L^p(0,T;B)$ to the problem of compactness with respect to the convergence in measure. Since this question is of independent interest, we are now going to present a characterization of relative compactness for the latter topology.

Uniform integrability and convergence in measure.
Let us denote by $\mathcal{M}(0,T;B)$ the vector space of strongly measurable B-valued functions; we recall that a sequence $\{u_n\} \subset \mathcal{M}(0,T;B)$ converges in measure to $u \in \mathcal{M}(0,T;B)$ if

$$\lim_{n\to\infty} |\{t \in (0,T) : \|u_n(t) - u(t)\|_B \ge \sigma\}| = 0 \quad \forall \sigma > 0$$

(where $|\cdot|$ denotes the Lebesgue measure on $(0,T)$). It is well known that $\mathcal{M}(0,T;B)$ can be endowed with a bounded distance which induces this convergence (see [6, III.2]), so that $\mathcal{M}(0,T;B)$ is an F-space (see [6, IV.11]); by the Chebychev inequality, it is easy to see that L^p convergence (compactness) yields convergence (compactness) in $\mathcal{M}(0,T;B)$. On the other hand, the latter notion, though weaker, entails the stronger one if some extra information of uniform integrability type is supplied.

We recall that a subset $\mathcal{U} \subset L^p(0,T;B)$ is *p-uniformly integrable* (or simply *uniformly integrable* if $p=1$) if

$$\forall \varepsilon > 0 \; \exists \delta > 0 : \; \forall I \subset (0,T) \quad |I| < \delta \Rightarrow \sup_{u \in \mathcal{U}} \int_I \|u(t)\|_B^p dt < \varepsilon$$

or, equivalently, if

$$\lim_{|I| \downarrow 0} \sup_{u \in \mathcal{U}} \int_I \|u(t)\|_B^p dt = 0 \tag{11}$$

Clearly the uniform integrability implies boundedness in $L^1(0,T;B)$; furthermore, when $B = \mathbf{R}$, it is equivalent to weak compactness in the space $L^1(0,T)$, (as a matter of fact, due to a lack of reflexivity, boundedness in the L^1 norm is no longer sufficient to ensure the existence of a subsequence converging in the weak topology of L^1), as it is stated by this fundamental

result:
Dunford-Pettis Criterion *Let $\mathcal{U} \subset L^1(0,T)$. The following conditions are equivalent:*

1. *$\mathcal{U}$ is (sequentially) weakly relatively compact;*

2. *$\mathcal{U}$ is uniformly integrable;*

3. *there exists a positive, convex and superlinearly increasing function $G : [0,+\infty) \to \mathbf{R}$ such that $\lim_{t\to\infty} \frac{G(t)}{t} = +\infty$ and*

$$\sup_{u \in \mathcal{U}} \int_0^T G(|u(t)|)dt < \infty$$

(See [5, Th. 22, 25 Chap.III], [6, Cor. IV.8.11] and [7, Th. 4.21.2] for the proof).

Remark 1.2 In particular, the previous characterization shows that boundedness in $L^p(0,T;B)$ entails q-uniform integrability for every $q < p$.

The link between L^p convergence, uniform integrability and convergence in measure is precised in [6, Th.III.3.6]:

Theorem 1.3 *On p-uniformly integrable sets the topologies of* $L^p(0,T;B)$ *and of* $\mathcal{M}(0,T;B)$ *coincide. In particular, a set* $\mathcal{U} \subset L^p(0,T;B)$ *is (relatively) compact in* $L^p(0,T;B)$ *iff it is p-uniformly integrable and (relatively) compact in* $\mathcal{M}(0,T;B)$.

A characterization of compactness in $\mathcal{M}(0,T;B)$**.**
We present now our second result on compactness for the convergence in measure; we set

$$\mathcal{D}(\mathcal{F}_t) := \{v \in B \ : \ \mathcal{F}(t,v) < +\infty\}.$$

Theorem 1.4 *A subset* $\mathcal{U} \subset \mathcal{M}(0,T;B)$ *is relatively compact with respect to the convergence in measure if and only if there exist* $\mathcal{F} : (0,T) \times B \to [0,+\infty)$ *and a lower-semicontinuous map* $g : B \times B \to [0,+\infty)$ *such that*

i) $\mathcal{F}$ *is a normal coercive integrand, (i.e. (7), (8) and (9) hold), and* $\mathcal{U}$ *is tight w.r.t.* $\mathcal{F}$*, i.e.*

$$\sup_{u \in \mathcal{U}} \int_0^T \mathcal{F}(t,u(t))dt < +\infty,$$

ii)

$$\lim_{h \to 0} \sup_{u \in \mathcal{U}} \int_0^{T-h} g(u(t+h),u(t))dt = 0, \tag{12}$$

iii)

$$\text{for a.e.} t \in (0,T) \quad \forall u,v \in \mathcal{D}(\mathcal{F}_t) \qquad g(u,v) = 0 \Rightarrow u = v \tag{13}$$

Moreover, if $\mathcal{U}$ *is relatively compact in measure, it is possible to choose* g *such that* g *is a continuous distance on* B*, i.e.* g *induces a weaker topology than the strong one.*

Remark 1.5 **i)** Thanks to Theorem 1.3, Theorem 1.1 is a consequence of the previous result, since if $\mathcal{U} \subset L^p(0,T;B)$ is bounded in the L^p norm and fulfils (1), $\mathcal{U}$ is also p- uniformly integrable.
In fact, we have

$$\lim_{h \to 0} \sup_{u \in \mathcal{U}} \int_0^{T-h} \Big| \|u(t+h)\|_B - \|u(t)\|_B \Big|^p dt = 0$$

which, together with the fact that $\{\|u\|_B \ : u \in \mathcal{U}\} \subset L^p(0,T)$ is bounded, implies, by the Riesz-Fréchet-Kolmogorov criterion, that $\{\|u\|_B\}_{u \in \mathcal{U}}$ is relatively compact in the strong topology of $L^p(0,T)$. Then it is easy to see that $\{\|u|_B^p\}_{u\in\mathcal{U}}$ is sequentially compact in the weak topology of $L^1(0,T)$, hence uniformly integrable by the Dunford-Pettis criterion.

ii) In accordance with what we have already remarked, in case B is continuosly embedded in another Banach space C, it is possible to choose

$$g(u,v) := \|u-v\|_C \quad \forall u,v \in B$$

2 Applications

Partial compactness.

Proposition 2.1 *Let $\mathcal{U} \subset L^p(0,T;B)$, $1 \le p < \infty$, be tight (with associated functional $\mathcal{F}$) and suppose that $[\,\cdot\,]$ is a l.s.c. seminorm on B such that*

$$\{[u]\ : u \in \mathcal{U}\} \quad \textit{is bounded in } L^p(0,T), \tag{14}$$

$$\lim_{h\downarrow 0}\sup_{u\in\mathcal{U}}\int_0^{T-h}[u(t+h)-u(t)]^p dt = 0, \tag{15}$$

and

$$\textit{for a.e. } t \quad [v-w]=0 \Rightarrow v=w \quad \forall v,w \in \mathcal{D}(\mathcal{F}_\sqcup) \tag{16}$$

If for $1 \le q \le p$

$$\forall \varepsilon > 0\ \exists C_\varepsilon > 0 :\ \forall t \in (0,T),\ \forall v \in B \quad \|v\|_B^q \le \varepsilon\mathcal{F}(t,v) + C_\varepsilon[v]^p \tag{17}$$

then $\mathcal{U}$ is compact in $L^q(0,T;B)$.

Remark 2.2 Let A, B, C be Banach spaces such that

$$A \subset B \quad \text{with compact injection} \tag{18}$$

and

$$B \subset C \quad \text{with continuous injection.} \tag{19}$$

J.-L. Lions (see [8, Lemma 5.1]) has proved that in this framework the following estimate holds:

$$\forall \varepsilon > 0\ \exists C_\varepsilon > 0 :\ \forall v \in A \quad \|v\|_B \le \varepsilon\|v\|_A + C_\varepsilon\|v\|_C \tag{20}$$

Hence, if we define $\mathcal{F}$ as in (5), it is clear that (17) holds with $[\,\cdot\,] := \|\cdot\|_C$ and $p=q$; thus if $\mathcal{U}$ is bounded in $L^p(0,T;A)$ and p-uniformly integrable in

$L^p(0,T;C)$, it is easy to see that $\mathcal{U}$ is p-uniformly integrable in $L^p(0,T;B)$, too.
In particular, if $\mathcal{U}$ is bounded in $L^p(0,T;A)\cap W^{1,p}(0,T;C)$, we conclude that $\mathcal{U}$ is relatively compact in $L^p(0,T;B)$ and retrieve Aubin's result.

A compactness result in the framework of the Stefan problem with the Gibbs-Thomson law.
The following result is a straightforward corollary of Theorem 1.1:

Theorem 2.3 *Let B and C Banach spaces, $T : B \to C$ a continuous map and $\{u_n\} \subset \mathcal{M}(0,T;B)$ be tight (with respect to the functional $\mathcal{F}$). Suppose also that*

- *for a.e. $t \in (0,T)$ $\quad \mathcal{D}(\mathcal{F}_t) \subset B$ satisfies:*

$$\forall u, v \in \mathcal{D}(\mathcal{F}_t) \quad Tu = Tv \Rightarrow u = v \tag{21}$$

- *$\{Tu_n\}$ is compact in measure in $\mathcal{M}(0,T;C)$.*

Then there exist $u_\infty : (0,T) \to A$ and a susequence $\{u_{n_k}\}$ with $u_{n_k} \to u_\infty$ in $\mathcal{M}(0,T;A)$.

This theorem allows to solve the ensuing compactness problem, which arises in the mathematical modelling of phase transitions.

Problem 2.4

Let Q be the cylindrical region $\Omega \times (0,T)$, ($\Omega \subset \mathbf{R}^3$ is a Lipschitz domain representing the region where the solid-liquid transition occurs in the time interval $(0,T)$), let Σ be an open subset of $\partial\Omega \times (0,T)$ such that

$$\Sigma_t := \{x \in \partial\Omega \ : (x,t) \in \Sigma\} \neq \emptyset \quad \forall t \in (0,T)$$

and let us denote with $H^1_{\Sigma_t}(\Omega)$ the Hilbert space

$$\left\{\omega \in H^1(\Omega) \mid \omega|_{\Sigma_t} = 0\right\}$$

(where $|_{\Sigma_t}$ is the restriction on Σ_t of the usual trace operator on $\partial\Omega$). Let us suppose that the sequences $\{\vartheta_n\}$, $\{\chi_n\} \subset L^1(Q)$, approximating the temperature and the phase variable, fulfil $\forall n$

$$|\chi_n(x,t)| = 1 \quad \text{for a.e.}(x,t) \in Q \tag{22}$$

$$\vartheta_n(\,\cdot\,,t) \in H^1_{\Sigma_t}(\Omega) \quad \text{for a.e. } t \in (0,T) \tag{23}$$

$$\sup_n \left\{ \int_0^T \left(\int_\Omega |D\chi_n| \right) dt + \int_Q |\nabla_x \vartheta_n(x,t)|^2 dxdt \right\} = M < \infty \tag{24}$$

(where $\chi \in BV(Q) \mapsto \int_Q |D\chi|$ is the *total variation functional*, (see e.g [14, XI, 1]), and

$$\partial_t(\vartheta_n + \chi_n) \quad \text{is bounded in } L^2(0,T;H^{-1}(\Omega)) \tag{25}$$

Can we infer that $\{\vartheta_n\}$ and $\{\chi_n\}$ admit convergent subsequences in $L^1(Q)$?

Let us remark that (22) yields a non-convex constraint on $\{\chi_n\}$ which is not preserved by the weak convergence in L^p.

Problem 2.4 was first successfully tackled by LUCKHAUS in the framework of the Stefan-Gibbs-Thomson problem for adiabatic nucleation in the case $\Sigma := \partial\Omega \times (0,T)$ (see [10, Lemma 2] and [14, Lemma 3.2]): it should be noticed that the original proof of Luckhaus'statement relies on capacity type estimates for Sobolev functions which it would be difficult to extend to the present situation, in which the boundary conditions depend on time.
We are going to show that Theorem 2.3 can be applied to obtain the compactness of the sequence $\{(\vartheta_n, \chi_n)\}$ even in this more general case.

First of all, we can see without effort that the choices $B := L^1(\Omega) \times L^1(\Omega)$, $(u := (\vartheta, \chi) \in B)$, $C := H^{-1}(\Omega)$ and

$$T(\vartheta, \chi) := \vartheta + \chi \tag{26}$$

$$\mathcal{F}(t,u) := \mathcal{F}_1(t,\vartheta) + \mathcal{F}_2(\chi) \tag{27}$$

with

$$\mathcal{F}_1(t,\vartheta) := \begin{cases} \int_\Omega |\nabla\vartheta(x)|^2 dx & \text{if } \vartheta \in H^1_{\Sigma_t}(\Omega) \\ +\infty & \text{otherwise} \end{cases}$$

$$\mathcal{F}_2(\chi) := \begin{cases} \int_\Omega |D\chi| & \text{if } \chi \in K := \{\chi \in L^1(\Omega) \mid \text{for a.e.} x \in \Omega \ |\chi(x)| = 1\} \\ +\infty & \text{otherwise} \end{cases}$$

yield Luckhaus's hypotheses.

If we rephrase Problem 2.4 in this way, Theorem 2.3 enables us to prove Proposition 2.5, which generalizes previous partial results by SAVARÉ [12] and PLOTNIKOV [11].

Proposition 2.5 *If $\{\vartheta_n\}$, $\{\chi_n\} \subset L^1(Q) = L^1(0,T;L^1(\Omega))$ fulfil (22, 23, 24, 25), there exist ϑ, $\chi \in L^1(Q)$ and subsequences $\{\vartheta_{n_k}\}$, $\{\chi_{n_k}\}$ such that $\vartheta_{n_k} \to \vartheta$, $\chi_{n_k} \to \chi$ in $L^1(Q)$.*

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